NONLINEAR EIGENVALUE PROBLEMS FOR EVEN FUNCTIONALS

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Abstract. Let \( H \) be a Hilbert space and let \( g \in C^1(H; \mathbb{R}) \) be an even Fréchet differentiable functional with completely continuous derivative. We introduce maximin values \( \sigma_k(t) \) which are critical values of \( g \) restricted to the sphere

\[ S_t = \{ u \in H; \frac{1}{2}\|u\|^2 = t \}, \]

and show that the functions \( \sigma_k(t) \) have right and left derivatives and that \( \sigma_k' \) are eigenvalues of \( g' \), i.e. there exist \( u_k^\pm \in S_t \) such that

\[ g'(u_k^\pm) = \sigma_k'(t)u_k^\pm. \]

Applications of the result are given to semilinear elliptic equations.

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1. Introduction

In this paper we study the eigenvalue problem

\[
(1.1) \quad g'(u) = \lambda u,\]

where \( g \) is a Fréchet differentiable functional on a Hilbert space \( H \) such that \( g' \) is completely continuous. Eigenvalue problems of this type are studied in many papers and, and it is not possible to give a complete list of these publications here. Instead, we refer the reader to the monographs [3] and [5] for a selection of references. See also the papers [6], [7] and [8] where related problems are studied.

We assume that \( g'(u) \neq 0 \) whenever \( u \neq 0 \), \( g(u) = 0 \), and \( g(u) > 0 \) for all \( u \neq 0 \). With these assumptions (1.1) is the critical point equation for \( g \) on the ball

\[ B_t = \{ u \in H; \frac{1}{2}\|u\|^2 \leq t \}. \]

If \( g \) is even, we put

\[ \Phi_t^k = \{ A \subset B_t; A = -A, A \text{ is compact}, \gamma(A) \geq k \}. \]

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where $\gamma(A)$ is the Krasnoselskii genus of $A$. Then the values

$$\sigma_k(t) = \sup_{A \in \mathfrak{A}^k} \min_{u \in A} g(u)$$

are critical values of $g$ restricted to $S_t = \partial B_t$.

Our main theorem proves that under some additional hypotheses on $g$, the critical value function $\sigma_k$ has right and left derivatives, and $\sigma_{k+}'$ and $\sigma_{k-}'$ are eigenvalues of the problem (1.1), i.e. there exist $u_{k}^+$ and $u_{k}^-$ such that

$$g'(u_{k}^\pm) = \sigma_{k\pm}'(t)u_{k}^\pm.$$

The linear counterpart of (1.1) is when $g$ is a positive definite quadratic form, $g(u) = \frac{1}{2}(Au, u)$, $g'(u) = Au$, and equation (1.1) reads

(1.2) \hspace{1cm} Au = \lambda u.$$

Under the assumption that $A : H \to H$ is a self-adjoint compact linear operator, by the spectral theorem there are countably many eigenvalues of (1.2) satisfying $\lambda_k > 0$ and $\lambda_k \to 0$ as $k \to \infty$. By the Rayleigh–Ritz characterization of the eigenvalues,

$$\lambda_k = \sup_{V \subset H} \min_{u \in V} \frac{(Au, u)}{\|u\|^2}$$

It can be proved that $\lambda_k$ coincides with $\sigma_k(1)$, so that by homogeneity $\sigma_k(t)$ are rays originating from the origin. Thus the nonlinear case can be seen as a generalization of the spectral theory of linear selfadjoint compact operators, where we instead of rays get curves originating from the origin having right and left derivatives at each $t > 0$.

As a first example, let $H = \mathbb{R}^2$ and let $g : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$g(x, y) = \frac{1}{2}x^2 + \frac{1}{4}y^4.$$  

Then

$$\sigma_1(t) = \max_{x^2+y^2=2t} g(x, y).$$  

Moreover, it can be shown that

$$\sigma_2(t) = \min_{x^2+y^2=2t} g(x, y).$$

A calculation shows that

$$\sigma_1(t) = \begin{cases} t & \text{for } t \leq 1 \\ t^2 & \text{for } t \geq 1, \end{cases}$$

and

$$\sigma_2(t) = \begin{cases} t^2 & \text{for } t \leq 1/2 \\ t - \frac{1}{4} & \text{for } t \geq 1/2. \end{cases}$$

Note that $\sigma_i$ has right and left derivatives for $i = 1, 2$ and that $\sigma_1$ is not differentiable at $t = 1$. However, the right and left derivatives exist for all $t > 0$. Note that for $t > 1/2$, there are other critical values of $g$ restricted to $S_t$ than $\sigma_1(t)$ and $\sigma_2(t)$. Indeed, for $t \in (1/2, 1)$ it is easily seen that $c = t^2$ is a third critical value, and for $t > 1$, $c = t$ is a third critical value.

As an application of the result to functional spaces, consider the elliptic differential equation

$$-\lambda \Delta u = f(x, u)$$

on a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$. The function $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ is subject to the conditions

($f_1$) for any $x \in \Omega$ and $s \in \mathbb{R}$,

$$f(x, s) = -f(x, -s).$$
(f_2) there exists a constant \( C > 0 \) and a \( p \in (1, 2N/(N - 2)) \) such that
\[
|f_s(x, s)| \leq C(1 + |s|^{p-2}).
\]
(f_3) for all \( x \in \Omega \) and \( s \in \mathbb{R} \),
\[
sf(x, s) \geq 0,
\]
and \( f(x, s) = 0 \) if and only if \( s = 0 \).

Letting \( H = H^1_0(\Omega) \) with inner product
\[
(u, v) = \int_\Omega \nabla u(x) \cdot \nabla v(x) \, dx
\]
and
\[
g(u) = \int_\Omega F(x, u(x)) \, dx,
\]
where
\[
F(x, s) = \int_0^s f(x, r) \, dr,
\]
our main theorem proves the existence of curves \( \sigma_k(t) \) which at each point \( t > 0 \) are critical values of \( g \) restricted to the sphere
\[
\frac{1}{2} \int_\Omega |\nabla u(x)|^2 \, dx = t.
\]

Theorem 1 speaks of the existence of \( u_k^\pm \in S_t \) such that
\[
-\Delta u_k^\pm = \sigma_k^{\prime \pm}(t) f(x, u_k^\pm).
\]

2. Preliminaries and statement of the main result

Let \( H \) be a Hilbert space, and let \( g \in C^1(H, \mathbb{R}) \) be a Fréchet differentiable functional on \( H \) such that \( g' \) is completely continuous (i.e. \( g' \) maps weakly convergent to strongly convergent sequences) and uniformly continuous on bounded subsets of \( H \). We assume that

(A1) \( g(u) = g(-u) \)

(A2) \( g(0) = 0 \),

and

(A3) \( g'(u) \neq 0 \) if \( u \neq 0 \).

For a fixed \( t > 0 \), we want to study critical points of \( g \) restricted to the sphere
\[
S_t = \{ u \in H; \frac{1}{2}||u||^2 = t \}.
\]

In order to do so, we introduce the ball
\[
B_t = \{ u \in H; \frac{1}{2}||u||^2 \leq t \}.
\]

For such \( t \) and integers \( k \geq 1 \), we let
\[
\Phi_k^t = \{ A \subset B_t; A = -A, A \text{ is compact, and } \gamma(A) \geq k \},
\]
where \( \gamma(A) \) is the Krasnoselski genus of \( A \), defined by
\[
\gamma(A) = \min\{ k \geq 1; \text{there exists an odd continuous mapping } h : A \rightarrow \mathbb{R}^k \setminus \{0\} \}
\]
and \( \gamma(A) = +\infty \) if there does not exist such a mapping for any \( k \geq 1 \).

Define the critical value function
\[
\sigma_k(t) = \sup_{A \in \Phi_k^t} \min_{u \in A} g(u).
\]
As Theorem 1 will show, the $\sigma_k(t)$ are indeed critical values of $g$ restricted to $S_t$. Let $t > 0$ and $k \geq 1$ be fixed. We make two further assumptions, namely that

$$(A4) \quad \sigma_k(t) > 0,$$

and that

$$(A5) \quad \text{there exists } \delta > 0 \text{ such that the equation } g'(u) = \lambda u \text{ does not have any solutions with } \lambda > 0 \text{ for } u \in B_t \cap g^{-1}(\sigma_k(t), \sigma_k(t) + \delta).$$

Condition (A4) is satisfied if e.g. $g(u) > 0$ for all $u \neq 0$, and condition (A5) is always satisfied when $k = 1$, which is the case of the maximum.

For every $t > 0$ and positive integer $k$, we have the inequalities

$$0 \leq \cdots \leq \sigma_{k+1}(t) \leq \sigma_k(t) \leq \cdots \leq \sigma_1(t) \leq \infty.$$

For each $t > 0$, $\lim_{k \to \infty} \sigma_k(t) = 0$ (see [2] p. 365), and since $\sigma_k(t) > 0$ there are infinitely many critical values for each $t > 0$.

We introduce the following set of critical points of $g$ restricted to the sphere $S_t$ using Lagrange multipliers:

$$M^k_t = \{ u \in H; \frac{1}{2}||u||^2 = t, \; g(u) = \sigma_k(t), \; \text{and } g'(u) = \lambda u \text{ for some } \lambda > 0 \}$$

If $u \in M^k_t$ and $\lambda > 0$ is such that $g'(u) = \lambda u$, then it is easy to see that

$$\lambda = \frac{g'(u), u}{2t}.$$ 

The main result is the following:

**Theorem 1.** Let $t > 0$. Suppose that $g \in C^1(H, \mathbb{R})$ is an even functional such that $g'$ is completely continuous, and such that conditions (A1), (A2), (A3), (A4) and (A5) are satisfied. Then the functions $\sigma_k(t)$ have right and left derivatives, $\sigma'_k(t)$ and $\sigma''_k(t)$, and there exist $u_{k+}, u_{k-} \in M^k_t$ such that

$$(2.1) \quad g'(u_{k+}) = \sigma'_k(t)u_{k+}, \quad g'(u_{k-}) = \sigma'_k(t)u_{k-}.$$ 

The following two sections concerns the proof of this theorem. In [6], Tintarev studied a similar problem for the mountain pass value. However, a gap in Lemma 3.2 of that paper makes its main assertion unverified. To avoid this difficulty, we impose the non-degeneracy condition (A5). Since this condition is difficult to verify in examples, it would be valuable if it could be eliminated or replaced by another condition which is easier to check.

In Section 4 we will see that $\sigma'_k(t)$ and $\sigma''_k(t)$ can be described as maximin values related to eigenfunctions in certain compact subsets of $M^k_t$. In Section 5, we will see an application to a semilinear elliptic equation.

In the sequel we will think of $k$ as fixed, and consequently we write $\Phi_t$, $\sigma(t)$, and $M_t$ in place of $\Phi^k_t$, $\sigma_k(t)$, and $M^k_t$, respectively.

**Proposition 1.** $\sigma$ is locally Lipschitz continuous.

**Proof.** Let $0 < t_1 \leq t \leq s \leq t_2$ and let $\epsilon > 0$. By the definition of $\sigma(s)$, there exists $A \in \Phi_t$ such that

$$\min_{u \in A} g\left( \sqrt{\frac{s}{t}}u \right) \geq \sigma(s) - \epsilon.$$
It follows from the requirements on \( g \) and \( g' \) that \( g \) is uniformly Lipschitz continuous on bounded subsets of \( H \). Hence, there exists \( C > 0 \), depending only on \( t_2 \), such that

\[
\max_{u \in A} \left| g \left( \sqrt{\frac{s}{t}} u \right) - g(u) \right| \leq C \left( \sqrt{\frac{s}{t}} - 1 \right) \max_{u \in A} \| u \| \leq C \left( \sqrt{\frac{s}{t}} - 1 \right) \sqrt{2t}.
\]

Hence

\[
\sigma(t) \geq \min_{u \in A} g(u) \\
\geq \min_{u \in A} g \left( \sqrt{\frac{s}{t}} u \right) - C \sqrt{2t} \left( \sqrt{\frac{s}{t}} - 1 \right) \\
\geq \sigma(s) - C \sqrt{\frac{s}{t}} \sqrt{\frac{2t}{t + \sqrt{s}}} (s - t) - \epsilon \\
\geq \sigma(s) - \frac{C}{\sqrt{2t_1}} (s - t) - \epsilon.
\]

Since \( \epsilon > 0 \) was arbitrary and \( \sigma \) is increasing,

\[
0 \leq \sigma(s) - \sigma(t) \leq \frac{C}{\sqrt{2t_1}} (s - t),
\]

and so

\[
|\sigma(s) - \sigma(t)| \leq M |s - t|,
\]

where \( M > 0 \) is a constant depending only on \( t_1 \) and \( t_2 \).

**Proposition 2.** \( M_t \) is a compact subset of \( S_t \), and for every \( t > 0 \) and \( \epsilon > 0 \) there is a \( \delta > 0 \) such that if \( |s - t| < \delta \) then

\[
(2.2) \quad \max_{u \in A} |\sigma(s) - \sigma(t)| \leq M |s - t|,
\]

where \( M > 0 \) is a constant depending only on \( t_1 \) and \( t_2 \).

**Proof.** Let \( u_j \) be a sequence in \( M_t \). Then there is a weakly convergent subsequence \( u_j \to u_0 \).

We have

\[
(2.4) \quad g'(u_j) = \lambda_j u_j, \\
\lambda_j = \frac{(g'(u_j), u_j)}{2t}
\]

and by the complete continuity of \( g' \), \( g'(u_j) \to g'(u_0) \) and \( (g'(u_j), u_j) \to (g'(u_0), u_0) \) as \( j \to \infty \).

So the left and right hand side of (2.4) tends strongly to \( g'(u_0) \), and weakly to \( \lambda_0 u_0 \), respectively, where

\[
\lambda_0 = \frac{(g'(u_0), u_0)}{2t}.
\]

Consequently, these limits must be equal, and we have

\[
g'(u_0) = \lambda_0 u_0.
\]

Observe that since \( g' \) is completely continuous, \( g \) is weakly continuous, and so by (A4), \( g(u_0) = \sigma(t) > 0 \). Assumptions (A2) and (A3) together imply that \( \lambda_0 \neq 0 \), and so we have

\[
(2.5) \quad \frac{1}{\lambda_j} g'(u_j) \to \frac{1}{\lambda_0} g'(u_0) = u_0.
\]

This proves that \( M_t \) is compact.

If (2.3) is false, then there is an \( \epsilon > 0 \) and a sequence \( u_j \in M_{t_j} \), where \( t_j \to t \), such that

\[
(2.6) \quad \frac{1}{\lambda_j} g'(u_j) \to \frac{1}{\lambda_0} g'(u_0) = u_0.
\]

This proves that \( M_t \) is compact.
and

\[ \lambda_j = \frac{(g'(u_j), u_j)}{2t_j} \]

By the complete continuity of \( g' \),

\[ g'(u_j) \to g'(u_0) \]

and

\[ \lambda_j \to \lambda_0 = \frac{(g'(u_0), u_0)}{2t}, \]

so by (2.6),

\[ g'(u_0) = \lambda_0 u_0. \]

Finally the weak continuity of \( g \) and the monotonicity implies that

\[ g(u_0) = \sigma(t) > 0, \]

and by assumption (A3), \( \lambda_0 \neq 0 \), and so

\[ u_j = \frac{1}{\lambda_j} g'(u_j) \to \frac{1}{\lambda_0} g'(u_0) = u_0, \]

and \( u_0 \in M_t \), contradicting (2.5).

\[ \square \]

3. Deformation of approximating sets

The two deformation lemmas of this section are based on the following lemma.

**Lemma 1.** Suppose that \( X(u), Y(u) \), are odd continuous mappings from a symmetric subset \( B \) of a Hilbert space \( H \) into \( H \). Let \( \hat{B} \) be the set of \( u \in B \) such that \( X(u) \neq 0 \), and assume that \( Y(u) \neq 0 \) for all \( u \) in a closed symmetric subset \( Q_0 \) of \( \hat{B} \). Assume also that there exists a positive number \( \theta < 1 \) such that

\[ (X(u), Y(u)) \leq \theta \|X(u)\| \|Y(u)\|, \quad u \in Q_0. \]

Then for each \( \alpha < 1 - \theta \), there exists an odd locally Lipschitz map \( Z(u) \) of \( \hat{B} \) into \( H \) such that

\[ \|Z(u)\| \leq 1 \text{ for all } u \in \hat{B}, \]

\[ (Z(u), Y(u)) < 0 \text{ for } u \in Q_0 \]

and

\[ (Z(u), X(u)) \geq \alpha \|X(u)\| \text{ for } u \in \hat{B}. \]

**Proof.** In [4], Schechter and Tintarev showed the existence of a vector field \( Z \) satisfying all requirements in the lemma except oddness. Thus the result follows by replacing \( Z \) by

\[ \frac{Z(u) - Z(-u)}{2}. \]

\[ \square \]

**Lemma 2.** Let \( g \in C^1(H, \mathbb{R}) \) be a functional such that \( g' \) is completely continuous and uniformly continuous on bounded subsets of \( H \). Suppose that \( g \) satisfies conditions (A1), (A2), (A3) and (A4). For \( t > 0 \) fixed, let \( \epsilon > 0 \), and let \( N_\epsilon = M_t + B_\epsilon \) be an \( \epsilon \)-neighborhood of \( M_t \). Then there exists \( \eta(\tau, u) \in C([0, 1] \times B_t, B_t) \) and a constant \( \delta > 0 \) such that

(i) \( \eta(0, u) = u \) for all \( u \in H \),

(ii) \( \eta(\tau, u) = u \) if \( g(u) \notin [\sigma(t) - 2\delta, \sigma(t) + 2\delta] \) or if \( u \in N_{\epsilon/4} \),

(iii) \( \eta(\tau, u) = -\eta(\tau, -u) \) for all \( \tau \in [0, 1] \),

(iv) \( g(\eta(\tau, u)) \geq g(u) \) for all \( u \in B_t \) and all \( \tau \in [0, 1] \),

(v) \( g(\eta(1, u)) \geq \sigma(t) + \delta \) for \( u \in B_t \) such that \( g(u) \geq \sigma(t) - \delta \) and \( u \notin N_\epsilon \).
Proof. Let $X(u) = g'(u)$, $Y(u) = u$ and $\tilde{B} = B_t \setminus \{0\}$. By Proposition 2, there exists $\nu > 0$ such that $M_s \subseteq N_{\varepsilon/2}$ if $|s - t| < \nu$. Let

$$Q_0 = ((\tilde{B}_t \setminus B_{t - \nu}) \setminus N_{\varepsilon/4}) \cap g^{-1}[\sigma(t) - 2\delta, \sigma(t) + 2\delta].$$

Now we claim that if $\delta$ and $\nu$ are chosen small enough, there exists a $\theta \in (0, 1)$ such that

$$(g'(u), u) \leq \theta \|g'(u)\| \|u\|$$

for any $u \in Q_0$. Indeed, if not, there exists a sequence $u_j \in B_t \setminus N_\varepsilon$ and $\lambda_j > 0$ such that

$$g(u_j) \to \sigma(t),$$

$$g'(u_j) - \lambda_j u_j \to 0,$$

$$\frac{1}{2} \|u_j\|^2 \to t.$$

Since $Q_0$ is bounded, $u_j$ has a subsequence for which $u_j \to u_0$, and by the complete continuity, $g'(u_j) \to g'(u_0)$. It follows that also $\lambda_j$ is bounded, and so it has a subsequence which converges to $\lambda_0 \geq 0$. By (3.5),

$$g(u_0) = \sigma(t),$$

$$g'(u_0) = \lambda_0 u_0,$$

$$\frac{1}{2} \|u_0\|^2 = t.$$

By condition (A3), $\lambda_0 \neq 0$, and we see that $u_0 \in M_t$, which is a contradiction.

Thus Lemma 1 is applicable, and choosing $\alpha < 1 - \theta$, there exists an odd locally Lipschitz map $Z : B_t \setminus \{0\} \to H$ satisfying (3.2), (3.3) and (3.4).

Observe that by (A3) and since $g'$ is completely continuous, there exists $\mu > 0$ such that $\|g'(u)\| > \mu$ for any $u \in Q_0$. We make the additional requirement that $\delta < \min(\alpha \mu/2, \epsilon \mu/4)$.

Let $\chi_1 : H \to [0, 1]$ and $\chi_2 : H \to [0, 1]$ be two locally Lipschitz continuous maps such that

$$\chi_1(u) = 0 \text{ when } u \in N_{\varepsilon/4},$$

$$\chi_1(u) = 1 \text{ when } u \notin N_{\varepsilon/2},$$

$$\chi_2(u) = 0 \text{ when } g(u) \notin [\sigma(t) - 2\delta, \sigma(t) + 2\delta],$$

$$\chi_2(u) = 1 \text{ when } g(u) \in [\sigma(t) - \delta, \sigma(t) + \delta].$$

Let $\chi = \chi_1 \chi_2$, and consider the initial value problem

$$\frac{d\eta(\tau, u)}{d\tau} = \chi(\eta(\tau, u)) Z(\eta(\tau, u)),$$

$$\eta(0, u) = u,$$

where $Z$ is as in Lemma 1. By basic theorems for ordinary differential equations, for each $u \in H$, there is a unique continuous solution $\eta(\tau, u)$ for $\tau \in [0, \infty)$. Since

$$\frac{d}{d\tau} \frac{1}{2} \|\eta(0, u)\|^2 = \left( \frac{d\eta(0, u)}{d\tau}, u \right) = \chi(u)(Z(u, u)) \leq 0$$

for all $u \in S_t = \partial B_t$, it is clear that $\eta : [0, 1] \times B_t \to B_t$. It is also clear that (i) is true, and (ii) follows since $\chi(u) = 0$ if $u \in N_{\varepsilon/4}$ or if $|g(u) - \sigma(t)| \geq 2\delta$. By the oddness of $Z$, (iii) follows.

By Lemma 1 and the chain rule,

$$\frac{d}{d\tau} g(\eta(\tau, u)) = \left( g'(\eta(\tau, u)), \frac{d\eta}{d\tau}(\tau, u) \right)$$

$$= (g'(\eta(\tau, u)), \chi(\eta(\tau, u)) Z(\eta(\tau, u)))$$

$$\geq \alpha \chi(\eta(\tau, u)) \|g'(\eta(\tau, u))\| \geq 0,$$

which proves (iv).
For $c \in \mathbb{R}$, let $E_c$ be the set
\[ E_c = \{ u \in B_t; g(u) \geq c \}. \]
To prove (v), we note that by (iv) it is sufficient to prove that $g(\eta(1, u)) \geq \sigma(t) + \delta$ if $u \in (E_{\sigma(t) - \delta} \setminus E_{\sigma(t) + \delta}) \setminus N$. For such $u$, there is a $\tau_1 > 0$ such that $\eta(\tau, u)$ remains in the region $(E_{\sigma(t) - \delta} \setminus E_{\sigma(t) + \delta}) \setminus N_{\epsilon/2}$ for $\tau \in [0, \tau_1)$. Thus by Lemma 1,
\[ 2\delta \geq g(\eta(\tau_1, u)) - g(u) = \int_0^{\tau_1} \frac{d}{d\tau} g(\eta(\tau, u)) \, d\tau \]
(3.6)
\[ \geq \alpha \int_0^{\tau_1} ||g'(\eta(\tau, u))|| \, d\tau \geq \alpha \mu \int_0^{\tau_1} ||Z(\eta(\tau, u))|| \, d\tau \]
\[ \geq \alpha \mu \left| \int_0^{\tau_1} Z(\eta(\tau, u)) \, d\tau \right| = \alpha \mu \left| \int_0^{\tau_1} \frac{d\eta}{d\tau}(\tau, u) \, d\tau \right| \]
\[ = \alpha \mu \|\eta(\tau_1, u) - u\|. \]
Thus, by the choice of $\delta$,
\[ \|\eta(\tau_1, u) - u\| \leq \frac{2\delta}{\alpha \mu} < \frac{\epsilon}{2}. \]
The orbit $\eta(\tau, u)$ cannot enter $N_{\epsilon/2}$ because the inequalities in (3.6) hold for any $\tau_1 > 0$ such that $\eta(\tau, u)$ remains in the indicated set for $0 \leq \tau \leq \tau_1$. Consequently, the only way for $\eta(\tau, u)$ to leave the set $(E_{\sigma(t) - \delta} \setminus E_{\sigma(t) + \delta}) \setminus N_{\epsilon/2}$ is by entering $E_{\sigma(t) + \delta}$. To show that this in fact occurs for $\tau \in (0,1]$, we suppose that $\eta(\tau, u) \in (E_{\sigma(t) - \delta} \setminus E_{\sigma(t) + \delta}) \setminus N_{\epsilon/2}$ for all $\tau \in [0,1]$. Then
\[ 2\delta > g(\eta(1, u)) - g(u) = \int_0^1 \frac{d}{d\tau} g(\eta(\tau, u)) \, d\tau \geq \alpha \mu > 2\delta, \]
which is a contradiction. Thus (v) is proved. \hfill \Box

Our next deformation lemma differs from our previous one in the sense that the deformation now takes place in a small neighborhood of a point $v \in M_t$.

**Lemma 3.** Assume that $g \in C^1(H, \mathbb{R})$ is even and that $g'$ is completely continuous. Suppose that conditions (A1), (A2), (A3), (A4) and (A5) are satisfied. If $v \in M_t$ and $\epsilon > 0$ is small enough, there exists $\delta \in (0, \delta)$, where $\delta$ is defined in condition (A5), such that for any $\nu > 0$ there exists $\eta(\tau, u) \in C([0,1] \times B_t, B_t)$ such that
\begin{enumerate}
  
(i) $\eta(0, u) = u$ for any $u \in B_t$,
(ii) $\eta(\tau, u) = u$ when $u \notin B_{\epsilon}(v) \cup B_{\epsilon}(-v)$,
(iii) $\eta(\tau, u) = -\eta(\tau, -u)$, for any $\tau \in [0,1]$ and $u \in B_t$,
(iv) $g(\eta(\tau, u)) \geq g(u)$, for any $\tau \in [0,1]$ and $u \in B_t$,
(v) if $u \in B_{\epsilon/2}(v)$ is such that $g(u) \geq \sigma(t) + \nu$, then $g(\eta(1, u)) \geq \sigma(t) + \delta$.
\end{enumerate}

**Proof.** We apply Lemma 1 with $X(u) = g'(u), Y(u) = u, \tilde{B} = B_t \setminus \{0\}$ and
\[ Q_0 = \{ u \in B_t \cap (B_{\epsilon}(v) \cup B_{\epsilon}(-v)); g(u) \geq \sigma(t) + \nu \}. \]
Let $\epsilon$ be so small such that there exists $\mu > 0$ such that $\|g'(u)\| > \mu$ whenever $u \in B_{\epsilon}(v)$. By condition (A5), there exists $\theta \in (0,1)$, such that
\[ (g'(u), u) \leq \theta \|g'(u)\| \|u\| \]
if $u \in Q_0$. Let $\alpha < 1 - \theta$, $\delta < \min(\delta, \alpha \mu, \alpha \mu / 2)$, and let $Z$ be as in Lemma 1. Let $\chi : H \to [0,1]$ be locally Lipschitz and such that
\[ \chi(u) = 1 \text{ when } u \in B_{\epsilon}(v) \cup B_{\epsilon}(-v) \text{ and } g(u) \geq \sigma(t) + \nu, \]
\[ \chi(u) = 0 \text{ when } u \notin B_{2\epsilon}(v) \cup B_{2\epsilon}(-v) \text{ or } g(u) \leq \sigma(t) + \nu / 2. \]
Then the initial value problem
\[ \frac{d\eta(t, u)}{dt} = \chi(\eta(t, u))Z(\eta(t, u)), \]
\[ \eta(0, u) = u \]
has a unique solution \( \eta(t, u) \) defined for \( t \in [0, \infty) \). As in the proof of Lemma 2, \( \eta \in C([0, 1] \times B_t, B_t) \). Assertions (i), (ii) and (iii) follow easily. The proof of (iv) is the same as the proof in Lemma 2. Let \( u \in B_{t/2}(v) \) be such that \( g(u) \in \{ \sigma(t) + \nu, \sigma(t) + \delta \} \). Then there exists \( \tau_1 \) such that \( \eta(t, u) \) remains in \( g^{-1}[\sigma(t) + \nu, \sigma(t) + \delta] \cap B_t(v) \) for \( t \in [0, \tau_1] \). As in the proof of (v) of Lemma 2,
\[ \|\eta(\tau_1, u) - u\| \leq \frac{\delta}{\alpha\mu} < \frac{\epsilon}{2} \]
by the choice of \( \delta \). Hence the orbit \( \eta(t, u) \) does not leave \( B_t(v) \). To see that \( g(\eta(1, u)) \geq \sigma(t) + \delta \), we assume the opposite, i.e. that \( g(\eta(t, u)) < \sigma(t) + \delta \) for all \( t \in [0, 1] \). Then
\[ \delta > g(\eta(1, u)) - g(u) = \int_0^1 \frac{d}{d\tau}g(\eta(\tau, u)) d\tau \geq \alpha\mu > \delta. \]
We get a contradiction and (v) is proved. \( \square \)

4. Critical sets and calculations of the derivatives

**Definition 1.** Let \( \Sigma_t \) be the collection of all sets \( S \in M_t \) such that there exists a sequence \( A_j \in \Phi_t \) such that
\[ \min_{u \in A_j} g(u) \to \sigma(t) \]
as \( j \to \infty \) and
\[ S = \{ v \in M_t; \liminf_{j \to \infty} d(v, A_j) = 0 \}. \]
Clearly, the members of \( \Sigma_t \) are compact since they are closed subsets of the compact set \( M_t \).

**Proposition 3.** For \( t > 0 \), \( \Sigma_t \) is nonempty and does not contain the empty set.

**Proof.** Any maximizing sequence gives rise to a set \( S \in \Sigma_t \), so obviously \( \Sigma_t \) is nonempty.

Let \( A_j \in \Phi_t \) and \( S \subset M_t \) be as in Definition 1. Suppose that \( S = \emptyset \). Then there exists an \( \epsilon > 0 \) such that \( A_j \cap (M_t + B_{\epsilon}(0)) = \emptyset \) for any \( j \) sufficiently large. Let \( \eta \) and \( \delta \) be as in Lemma 2, and let \( \tilde{A}_j = \eta(1, A_j) \). Then by Lemma 2,
\[ \min_{u \in A_j} g(u) \geq \sigma(t) + \delta. \]
But this cannot be possible, since \( \tilde{A}_j \in \Phi_t \). \( \square \)

**Definition 2.** Let \( \Sigma_t^0 \) be the collection of all sets \( S_0 \subset M_t \) such that there exists a sequence \( A_j \in \Phi_t \) such that
\[ \min_{u \in A_j} g(u) \to \sigma(t) \]
and
\[ S_0 = \{ v \in M_t; \liminf_{j \to \infty} d(v, A_j) = 0 \text{ and for every } \epsilon > 0, \text{ there exists } j_0 \geq 1 \text{ such that for any } j \geq j_0, \inf_{u \in A_j \cap B_\epsilon(v)} g(u) \leq \sigma(t) \}. \]
If \( S_0 \in \Sigma_t^0 \), then \( S_0 \) is compact, and there exists a set \( S \in \Sigma_t \) such that \( S_0 \subset S \).

**Proposition 4.** The family \( \Sigma_t^0 \) is nonempty and does not contain the empty set.
Proof. By the same reason as for $\Sigma_\ell$, the class $\Sigma^0_\ell$ is nonempty.
Suppose that $S_0 \in \Sigma^0_\ell$ is empty. Then there exists $\epsilon > 0$ such that
$$\inf_{u \in A_j} g(u) > \sigma(t),$$
d(\epsilon, M_\ell) \leq \epsilon

Let $\eta$ and $\delta$ be as in Lemma 2, and let $\tilde{A}_j = \eta(1, A_j)$. Then
$$\min_{u \in \tilde{A}_j} g(u) \geq \sigma(t) + \delta,$$
and so
$$\min_{u \in \tilde{A}_j} g(u) > \sigma(t),$$
which contradicts the definition of $\sigma(t)$ since $\tilde{A}_j \in \Phi_\ell$. \qed

**Proposition 5.** If $S_0 \in \Sigma^0_\ell$, then for any $\alpha > 0$, there exists a set $S \in \Sigma_\ell$ such that $S_0 \subset S \subset S_\alpha + B_\alpha(0)$.

**Proof.** Let $S_0, A_j$ be as in Definition 2, and let $S \in \Sigma_\ell$ be obtained from the sequence $A_j$ as in Definition 1. Let $\epsilon > 0$ be given. If $v \in S$ is such that
$$\inf_{u \in A_j \cap B_\epsilon(v)} g(u) > \sigma(t),$$
then we can apply Lemma 3, and obtain a new sequence $\tilde{A}_j \in \Phi_\ell$ such that
$$\inf_{u \in A_j \cap B_\epsilon(v)} g(u) \geq \sigma(t) + \delta.$$

Let
$$S_\epsilon = \{v \in M_\ell; d(v, \tilde{A}_j) \to 0\},$$
and note that $S_\epsilon \in \Sigma_\ell$ and that $v \notin S_\epsilon$. In this way we remove points $v$ in $S$ such that (4.1) holds. Since $S$ is compact, all such points can be removed in finitely many steps. This means that for $v \in S_\epsilon$
$$\inf_{u \in A_j \cap B_\epsilon(v)} g(u) \leq \sigma(t)$$

holds.

Now this can be repeated for any $\epsilon > 0$, and we have $S_0 \subset S_\epsilon$. It is also easy to see that $\cap_{\epsilon > 0} S_\epsilon \subset S_0$. Therefore
$$S_0 = \bigcap_{\epsilon > 0} S_\epsilon,$$
so that $S_0$ is the intersection of a nested sequence of sets in $\Sigma_\ell$. But then it is clear that $S_0$ can be approximated from above by sets in $\Sigma_\ell$. \qed

**Lemma 4.**
$$\limsup_{s \to t^+} \frac{\sigma(s) - \sigma(t)}{s - t} \leq \frac{\sup_{S_0 \in \Sigma_\ell} \min_{v \in S_0} \left(\frac{\sigma'(v)}{2t}\right)}{s - t}.$$

**Proof.** Let $t_j \to t^+$, and let $A_j \in \Phi_\ell$ be such that
$$\sigma(t_j) = \min_{u \in A_j} g(u \sqrt{t_j \ell u}) + o(t_j - t).$$

Let $S_0$ be the set in $\Sigma^0_\ell$ obtained from the sequence $A_j$.

Let $v \in S_0$ be arbitrary, and let $\epsilon > 0$. Then $A_j$ has a subsequence for which $A_j \cap B_\epsilon(v) \neq \emptyset$. Thus
$$\sigma(t_j) = \min_{u \in A_j} g(u \sqrt{t_j \ell u}) + o(t_j - t) \leq \inf_{u \in A_j \cap B_\epsilon(v)} g(u \sqrt{t_j \ell u}) + o(t_j - t).$$
By the mean value theorem and the definition of $\Sigma^0_t$,\
\[
\sigma(t_j) \leq \inf_{u \in A_j \cap B_r(v)} g(u) + \sup_{u \in B_r(v)} \sup_{\theta \in [1, \sqrt{t_j}/t]} (g'(\theta u), u) \left( \frac{\sqrt{t_j}}{t} - 1 \right) + o(t_j - t)
\]
\[
\leq \sigma(t) + \sup_{u \in B_r(v)} \sup_{\theta \in [1, \sqrt{t_j}/t]} (g'(\theta u), u) \left( \frac{\sqrt{t_j}}{t} - 1 \right) + o(t_j - t).
\]
Hence
\[
\limsup_{j \to \infty} \frac{\sigma(t_j) - \sigma(t)}{t_j - t} \leq \sup_{u \in B_r(v)} \frac{(g'(v), v)}{2t},
\]
and since $\epsilon > 0$ was arbitrary,
\[
\limsup_{j \to \infty} \frac{\sigma(t_j) - \sigma(t)}{t_j - t} \leq \frac{(g'(v), v)}{2t}.
\]
Now the same argument is true for any $v \in S_0$, and so
\[
\limsup_{j \to \infty} \frac{\sigma(t_j) - \sigma(t)}{t_j - t} \leq \min_{v \in S_0} \frac{(g'(v), v)}{2t},
\]
and (4.2) follows since $t_j \to t+$ was arbitrary. \qed

**Lemma 5.**

\[
(4.3) \quad \liminf_{s \to t} \frac{\sigma(s) - \sigma(t)}{s - t} \geq \sup_{S \in \Sigma^0 \cap S_0} \min_{v \in S_0} \frac{(g'(v), v)}{2t}.
\]

**Proof.** Let $t_j \to t+$, and let $S \in \Sigma_t$ be arbitrary. By the definition of $S$, there is a sequence $A_j \in \Phi_t$, such that
\[
\min_{u \in A_j} g(u) \to \sigma(t)
\]
and
\[
S = \{ v \in M_t; d(v, \tilde{A}_t) \to 0 \}.
\]
Let $\epsilon > 0$ and let $\eta$ be as i Lemma 2. Put $A_t = \eta(1, \tilde{A}_t)$. It is clear that $A_t \in \Phi_t$ and that
\[
\min_{u \in A_t} g(u) \to \sigma(t)
\]
as $l \to \infty$. By the proof of Lemma 2 it follows that $\|\eta(1, u) - u\| \leq \epsilon/2$. Hence if $\epsilon$ is sufficiently small, then by (v) of Lemma 2
\[
\min_{u \in A_t} g(u) \geq \sigma(t) + \delta.
\]
for $l$ sufficiently large.

Since $g$ is uniformly continuous on bounded sets, there exists $j_0 \geq 1$ (independent of $l$) such that for all $j \geq j_0$
\[
\min_{u \in A_j} g \left( \sqrt{\frac{t_j}{t}} u \right) = \min_{u \in A_j; \|u\| \leq \epsilon} g \left( \sqrt{\frac{t_j}{t}} u \right).
\]
By the mean value theorem,
\[
\sigma(t_j) \geq \min_{u \in A_t} \frac{g(u)}{d(u,S)} \left( \sqrt{\frac{t_j}{t}} - 1 \right)
\]
(4.4)
\[
\geq \min_{u \in A_t} \frac{g(u)}{d(u,S)} + \inf_{\theta \in [1, \sqrt{t_j/t}]} \left( g'(\theta u), u \right) \frac{t_j - t}{\sqrt{t_j} + \sqrt{t}}.
\]

Letting \( l \to \infty \),
\[
\sigma(t_j) \geq \sigma(t) + \inf_{\theta \in [1, \sqrt{t_j/t}]} \frac{g'(\theta u), u}{\sqrt{t_j} + \sqrt{t}} (t_j - t).
\]
Thus
\[
\lim \inf_{j \to \infty} \frac{\sigma(t_j) - \sigma(t)}{t_j - t} \geq \inf_{d(u,S) \leq \epsilon} \frac{g'(u), u}{2t}.
\]
Since \( \epsilon \) was arbitrary,
\[
\lim \inf_{j \to \infty} \frac{\sigma(t_j) - \sigma(t)}{t_j - t} \geq \min_{v \in S} \frac{g'(v), v}{2t}.
\]
Let \( S_0 \in \Sigma^0_t \) be arbitrary. By Proposition 5, for any \( \alpha > 0 \), there exists \( S \in \Sigma_t \) such that \( S_0 \subset S \subset S_0 + B_\alpha \). Thus
\[
\lim \inf_{j \to \infty} \frac{\sigma(t_j) - \sigma(t)}{t_j - t} \geq \min_{v \in S_0} \frac{g'(v), v}{2t},
\]
and since \( S_0 \) and \( t_j \) was arbitrary,
\[
\lim \inf_{s \to t^+} \frac{\sigma(s) - \sigma(t)}{s - t} \geq \sup_{S_0 \in \Sigma^0_t} \min_{v \in S_0} \frac{g'(v), v}{2t}.
\]

\( \square \)

**Proof of Theorem 1.** By Lemma 4 and Lemma 5 we have that
\[
\sup_{S_0 \in \Sigma^0_t} \min_{v \in S_0} \frac{g'(v), v}{2t} \leq \lim \inf_{s \to t^+} \frac{\sigma(s) - \sigma(t)}{s - t} \leq \sup_{S_0 \in \Sigma^0_t} \min_{v \in S_0} \frac{g'(v), v}{2t}.
\]
Thus the right derivative \( \sigma'_+(t) \) exists and is given by
\[
\sigma'_+(t) = \sup_{S_0 \in \Sigma^0_t} \min_{v \in S_0} \frac{g'(v), v}{2t}.
\]
A similar argument shows that \( \sigma'_-(t) \) exists and is given by
\[
\sigma'_-(t) = \inf_{S_0 \in \Sigma^0_t} \max_{v \in S_0} \frac{g'(v), v}{2t}.
\]
To prove (2.1), note that by (4.5), there exists a sequence \( v_j \in M_t \) such that
\[
\frac{g'(v_j), v_j}{2t} \to \sigma'_+(t).
\]
Since $M_t$ is compact, $v_j$ has a convergent subsequence $v_j \to v_0 \in M_t$. Thus there exists $\lambda_0 > 0$ such that
\[ g'(v_0) = \lambda_0 v_0. \]
But we also have
\[ \lambda_0 = \frac{(g'(v_0), v_0)}{2t} = \lim_{j \to \infty} \frac{(g'(v_j), v_j)}{2t} = \sigma'_+(t). \]
This proves the first part of (2.1). The second part is proved by a similar argument using (4.6) instead of (4.5).

5. Applications to semilinear elliptic problems

We return to the elliptic partial differential equation mentioned in the introduction, i.e. let $\Omega$ be a bounded domain in $\mathbb{R}^N$ ($N \geq 3$) and consider the problem
\[ -\lambda \Delta u = f(x, u), \quad u \in H^1_0(\Omega). \]
Here $H^1_0(\Omega)$ denotes the Sobolev space $W^{1,2}_0(\Omega)$ which is the completion of $C_0^\infty(\Omega)$ in the norm $\|u\|$ defined by
\[ \|u\|^2 = \int_\Omega |\nabla u(x)|^2 \, dx. \]
The function $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ is subject to the conditions
\begin{enumerate}[(f_1)]  \item $f$ is odd in $u$, i.e. for every $x \in \Omega$ and $u \in \mathbb{R}$, $f(x, u) = -f(x, -u)$,
  \item there exists a constant $C > 0$ and $p \in (1, 2N/(N-2))$ such that for all $x \in \Omega$ and $u \in \mathbb{R}$,
    \[ |f_u(x, u)| \leq C \left( 1 + |u|^{p-2} \right), \]
  \item for all $x \in \Omega$ and $u \in \mathbb{R}$,
    \[ uf(x, u) \geq 0, \]
    and $f(x, u) = 0$ if and only if $u = 0$.
\end{enumerate}
The space $H^1_0(\Omega)$ is a Hilbert space with the inner product
\[ (u, v) = \int_\Omega \nabla u(x) \cdot \nabla v(x) \, dx. \]
Let
\[ F(x, u) = \int_0^u f(x, s) \, ds, \]
and consider the functional
\[ g(u) = \int_\Omega F(x, u(x)) \, dx. \]
To apply Theorem 1, we need to prove that $g \in C^1(H^1_0(\Omega), \mathbb{R})$ satisfies that $g'$ is completely continuous and uniformly continuous on bounded subsets of $H^1_0(\Omega)$, and that conditions (A1)-(A5) are satisfied.
We will see that $g$ is in fact two times differentiable, and that $g''$ is completely continuous. This implies that $g'$ is uniformly Lipschitz continuous on bounded subsets of $H^1_0(\Omega)$. Indeed, if $g''$ is completely continuous, it is bounded (it is actually compact). Let $B_R$ be a ball of radius $R$ in $H$. If $u, v \in B_R$, then by the mean value theorem,
\[ \|g'(u) - g'(v)\| \leq \max_{t \in [0,1]} \|g''(v + t(u - v))\| \|u - v\| \leq C_R \|u - v\| \]
for some constant $C_R > 0$. 
Formally,
\[ \langle g'(u), v \rangle = \int_\Omega f(x, u(x))v(x) \, dx. \]

By the Sobolev inequality (see e.g. [1]), \( H^1_0(\Omega) \subset L^q(\Omega) \) for \( q \in [1, 2N/(N - 2)] \), and the inclusion is continuous, i.e.
\[ \|u\|_{L^q} \leq C_q \|u\|_{H^1_0} \]
for all \( u \in H^1_0(\Omega) \). Moreover, since \( \Omega \) is a bounded domain, the inclusion is compact for \( q \in [1, 2N/(N - 2)] \).

Condition (f2) and the Lebesgue dominated convergence theorem together imply that \( g'(u) \) exists for all \( u \in H^1_0(\Omega) \).

To see that \( g' \) is completely continuous, note that
\[ \|g'(u)\| = \sup_{0 \neq v \in H^1_0(\Omega)} \frac{|\int_\Omega f(x, u(x))v(x) \, dx|}{\|v\|}. \]

Then by the Hölder and Sobolev inequalities,
\[ \|g'(u)\| \leq C \left( \int_\Omega |f(x, u(x))|^{2N/(N+2)} \, dx \right)^{(N+2)/2N}. \]

Let \( u_m \to u \). By (5.2), it suffices to show that
\[ \int_\Omega |f(x, u_m(x)) - f(x, u(x))|^{2N/(N+2)} \, dx \to 0 \]
as \( m \to \infty \).

A theorem of Krasnoselskii [2] states that if \( h \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R}) \) and
\[ |h(x, u)| \leq C(1 + |u|)^{\gamma/\delta}, \]
where \( 1 \leq \gamma, \delta < \infty \), then the map \( w(x) \mapsto h(x, w(x)) \) is continuous from \( L^\gamma(\Omega) \) to \( L^\delta(\Omega) \). We apply this theorem with \( h = f, \delta = 2N/(N+2) \) and \( \gamma = \delta p \). Thus, we see that (5.3) is satisfied if \( u_m \to u \) in \( L^p(\Omega) \). But this follows by the compactness of the Sobolev embedding. We have proved that \( g' \) is completely continuous.

It remains to check that \( g'' \) exists and is completely continuous. Formally,
\[ \langle g''(u)v, w \rangle = \int_\Omega f_u(x, u(x))v(x)w(x) \, dx. \]

Repeating the argument for \( g' \) shows that \( g'' \) is completely continuous.

It is easy to see that \( g \) is even and that \( g(0) = 0 \), so obviously (A1) and (A2) are valid. Condition (A3) and (A4) follows by \( (f_3) \).

Condition (A5) is difficult to verify in practice, and it is not clear if it is valid for \( \sigma_k, k \geq 1 \).

However, when \( k = 1 \), \( \sigma_1(t) = \sup_{u \in S_1} g(u) \), so obviously, (A5) is satisfied for \( k = 1 \). It is also satisfied when \( f(u) = u \), by the theory of compact linear operators. By application of Theorem 1, the right and left derivatives of \( \sigma_1(t) \) exist, and there are \( u_1^+ \) and \( u_1^- \) in \( S_1 \) such that
\[ -\Delta u_1^\pm = \sigma_1' \pm f(x, u_1^\pm). \]

Of course, \( \sigma_1'(+) \) and \( \sigma_1'(-) \) may coincide, which in fact happens for almost every \( t \).

Let us conclude with a situation where \( \sigma_k(t) \) can be calculated explicitly. Let
\[ f(u) = |u|^{p-2}u, \]
where as before, \( p \in [2, 2N/(N-2)] \) and \( N \geq 3 \). It is clear that \( f \) satisfies the conditions (\( f_1 \))-(\( f_3 \)) stated above. Thus, the functional
\[ g(u) = \int_\Omega |u(x)|^p \, dx \]
belongs to $C^1(H^1_0(\Omega), \mathbb{R})$, and conditions (A1)-(A4) are valid. Moreover, the homogeneity of $g$ makes it possible to calculate
\[ \sigma_k(s) = \left( \frac{s}{t} \right)^{p/2} \sigma(t). \]
In particular,\[ \sigma_k(t) = t^{p/2} \sigma_k(1), \]
and so $\sigma_k$ is indeed differentiable for every $t > 0$, and\[ \sigma_k'(t) = \frac{p}{2} t^{p/2-1} \sigma_k(1) = t^{p/2-1} \sigma_k'(1). \]

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