Shape Optimization for Incompressible Laminar Flows.

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Abstract

The problem to minimize the drag on a body that moves in an incompressible fluid is considered. We show that the difference between two solutions is small if the difference between the domains of the flow is small. This is then used to show that the energy functional describing the drag is continuous on a compact set, which implies that there exists a minima. An attempt to find the minima is then made by using steepest decent methods in MATLAB and FEMLAB.

Preface

This masters thesis was written in the Spring of 2004 during my diploma project, which terminates my undergraduate studies in Electrical Engineering at the Lund Institute of Technology.

A huge amount of appreciation goes to the following exalted members of the human race. My supervisor Magnus Fontes for his guidance, ideas and enthusiasm. Olivier Verdier for taking time to read my manuscript and offering valuable advice. Olof Barr and Henrik Stewenius for allowing me to share their office and for general support. Anders Holst and Sven Spanne for valuable help on the subject. Finally I would like to thank the staff at the Centre for Mathematics for stimulating discussions during, sometimes very long, coffee brakes.

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Chapter 1

Introduction

1.1 Background

In the spring of 2003 Timmy Fagerlund made a study (see [6]) of how the shape of some birds change before they are going to migrate. In the study he took MRT (magnetic resonance tomography) pictures of birds from four different species and measured how the fat was deployed in the body. The study showed that the fat is not deposited evenly over the body as one might assume, it was in fact localized to the front and the rear of the body. A reason for this might be that the birds try to minimize the maximum cross-section area in order to minimize the drag. In this Master's thesis we will try to find out if there is such a minimum, and if so, try to find a numerical approximation of it.

1.2 Nomenclature

In order to avoid confusion we will, throughout this thesis, use the following notation:

- We denote the identity map Id and the identity matrix I. Note that I is the functional matrix of Id.
- Ψ will always be the inverse of Φ.
- If $\Phi : \Omega \to \tilde{\Omega}$ where $\Omega, \tilde{\Omega} \subseteq \mathbb{R}^n$, then Φ' is the functional matrix of Φ. Thus, $\det \Phi'$ will be the Jacobian of Φ.
- If $\Phi : \Omega \to \tilde{\Omega}$ where $\Omega, \tilde{\Omega} \subseteq \mathbb{R}^n$ then

$$(\partial_i \Phi) = \begin{pmatrix} \partial_i \Phi_1 \\ \partial_i \Phi_2 \\ \vdots \\ \partial_i \Phi_n \end{pmatrix}.$$

- If $\Phi : \Omega \to \tilde{\Omega}$ where $\Omega \subseteq \mathbb{R}^n$, $\tilde{\Omega} \subseteq \mathbb{R}^m$ then

$$(D\Phi) = \begin{pmatrix} \partial_1\Phi_1 & \partial_2\Phi_1 & \dots & \partial_n\Phi_1 \\ \partial_1\Phi_2 & \partial_2\Phi_2 & \dots & \partial_n\Phi_2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1\Phi_m & \partial_2\Phi_m & \dots & \partial_n\Phi_m \end{pmatrix}.$$

- If $u : \Omega \to \mathbb{R}$ where $\Omega \subseteq \mathbb{R}^n$ then

$$\nabla u = \begin{pmatrix} \partial_1 u \\ \partial_2 u \\ \vdots \\ \partial_n u \end{pmatrix}.$$

  Note that $\nabla u = (Du)^T$.

- If $u : \Omega \to \mathbb{R}$ where $\Omega \subseteq \mathbb{R}^n$ then D^2u is the Hessian, that is

$$(D^2u) = \begin{pmatrix} \partial_{11}u & \partial_{12}u & \dots & \partial_{1n}u \\ \partial_{21}u & \partial_{22}u & \dots & \partial_{2n}u \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{n1}u & \partial_{n2}u & \dots & \partial_{nn}u \end{pmatrix}.$$

- If $u : \Omega \to \tilde{\Omega}$ where $\Omega \subseteq \mathbb{R}^n$, $\tilde{\Omega} \subseteq \mathbb{R}^m$ and $\Phi : S \to \Omega$ where $S \subset \mathbb{R}^k$ then

$$u \circ \Phi = u(\Phi(x))$$

Also, due to the difficulty of naming new constants, we will keep the same symbol for the constants (mostly c) even if they have changed from one line to the next. We often use functions in the Sobolev spaces $W^{s,\infty}$, strictly speaking it does not make sense to talk about pointwise values of these functions. However it is well known that any function in $W^{s,\infty}$ is equal to a function in $C^{s-1,1}$ everywhere except on a set of measure zero. That is, if $\Phi \in W^{s,\infty}$ then Φ can be represented by a function whose derivatives of order 0 to $s-1$ are Lipschitz continuous. Therefore, when we speak of pointwise values it is understood that this is the values of the representative in $C^{s-1,1}$. In order to make the presentation, short most of the theoretical background and the more trivial parts of the thesis have been put in an appendix. Readers who are not familiar with e.g L^p and Sobolev spaces are encouraged to start by reading the Appendix.

1.3 Problem definition

We will try to model the flight of the bird by putting an obstacle in an air flow and determine the drag. We take U to be an open bounded set (in $\mathbb{R}^2$ or $\mathbb{R}^3$) and in the center we take a closed set B. In figure 1.3 we have drawn U as a rectangle, however most of the theorems require that the boundary of U is e.g. Lipschitz continuous, therefore we will say that U is diffeomorphic to an ellipse (in $\mathbb{R}^2$ and a sphere in $\mathbb{R}^3$). The set U should be thought of as the universe in which the bird lives in, that is a portion of the sky. B will be a nice set in the sense that it has properties like simply connected and twice differentiable boundary (in the numerical part B will be an ellipse). The set B should be thought of as the bird. The domain in which the flow u will be in is then $D = U \setminus B$. These sets will be kept constant throughout the thesis.

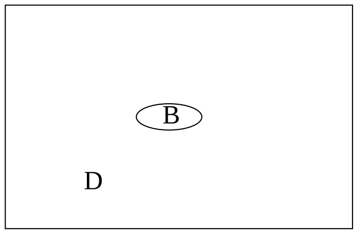

Figure 1.1: The domain U.

In order to compare different shapes of birds we will say that there are mappings $\Phi : U \to U$ that maps the sets B and D to $\Phi(B)$ and $\Phi(D)$. The new shape of the bird is then $\Phi(B)$ and the flow is in $\Phi(D)$. We will always deal with simple mappings such that the boundary $\partial\Phi(D)$ of $\Phi(D)$ will be equal to $\Phi(\partial D)$. This is clear if e.g. $\Phi = Id$ close to ∂U and Φ is a 2-diffeomorphism (see Appendix, Definition A.5).

The flow is modelled by the steady state Navier-Stokes equations, which are

$$\begin{cases} -\nu\Delta u_\Phi + \sum_{i=1}^n u_{\Phi i}\partial_i u_\Phi + \nabla p_\Phi = 0 & in\ \Phi(D) \\ \text{div}\ u_\Phi = 0 & in\ \Phi(D) \\ u_\Phi = f \circ \Phi^{-1} & on\ \partial\Phi(D) \end{cases}$$

where u_Φ is the velocity of the fluid and p_Φ is the pressure. Note that u_Φ is a vector valued function.

1.3.1 The drag power

The forces on the body is made up of two terms, namely the viscous force $\nu\frac{\partial u}{\partial n}$ and the pressure forces $-pn$, where n is outer normal to the boundary ∂B. The total force on the body is

$$\int_{\partial B} (\nu\frac{\partial u}{\partial n} - pn)dx$$

Calculating the power of the drag is under certain conditions equivalent to calculating the energy integral

$$E = \int_D |\nabla u|^2\, dx$$

To see this we do the following: We let u_0 be constant $(k, 0, 0)$ (in $\mathbb{R}^3$ or $(k, 0)$ in $\mathbb{R}^2$). We let the boundary conditions be

$$\begin{cases} u = 0 & \text{on } \partial B \\ u = u_0 & \text{on } \partial D \setminus \partial B \end{cases}$$

If we now put $\phi = u - u_0$ and insert in the weak form (A.6) (see appendix, here we have kept the boundary terms, and put $f = 0$) we get

$$\nu \int_D \nabla u \cdot \nabla (u - u_0)\, dx + \int_D \sum_i u_i \partial_i u (u - u_0)\, dx = \int_{\partial D} (\nu \frac{\partial u}{\partial n} - pn)(u - u_0)\, dx$$

Since u_0 is constant we have that $\partial_i u = \partial_i (u - u_0)$, and due to Lemma A.4 the nonlinear term disappears. For the boundary terms we have that $u - u_0 = 0$ on $\partial D \setminus \partial B$ and $u = 0$ on ∂B. We also have that $\nabla u_0 = 0$. This gives us

$$\nu \int_D \nabla u \cdot \nabla u\, dx = -u_0 \int_{\partial B} (\nu \frac{\partial u}{\partial n} - pn)\, dx$$

This shows that the value of E is in fact proportional to the drag power.

To minimize the drag power is thus equivalent to minimizing the energy integral

$$E(\Phi) = \int_{\Phi(D)} |\nabla u_\Phi|^2\, dx.$$

Note that when u_Φ is a vector function we mean that $|\nabla u_\Phi|^2 = |\nabla u_{\Phi 1}|^2 + ... + |\nabla u_{\Phi n}|^2$. The integral $E(\Phi)$ will be our goal function which will be minimized with respect to Φ.

1.3.2 The full problem

In order to have a good way to measure differences between different domains we will say that a domain $\Phi(D)$ is close to D if the mapping Φ is close to the identity, in some sense. We only allow domains for which there are mappings that fulfills some special conditions, listed below.

Let $\mathcal{A}$ be the set of admissible mappings. Because we don't want the mappings to destroy some smoothness properties of the domains we only allow mappings which are in $W^{2,\infty}(U)$ (see Definition A.1 appendix). It can be shown that if $\Phi \in W^{2,\infty}(U)$ then there is a $\Phi_c \in C^{1,1}(U)$ such that Φ_c and Φ is equal everywhere except on a set of measure zero, that is we can always think of Φ as a function which is Lipschitz continuous and with Lipschitz continuous derivatives (see [13]).

Moreover we put a bound on the difference between the mapping and the identity, such that $||\Phi - Id||_{W^{2,\infty}(U)} \leq k$ where k is a constant. We also want some points in U to be fix points. We denote the set of fix points F.

We also want the mappings to be invertible. Therefore we demand that the set F will include all points which are close to the boundary of U and that

$$||\Phi' - I||_2 := \sup_{|v|_2=1} |(\Phi' - I)v|_2 \leq 1/2 \quad \forall x \in U \tag{1.1}$$

Here $|\cdot|_2$ denotes the absolute value in $\mathbb{R}^n$. This together with the fact that Φ can be chosen to be continuous, will be enough to prove that Φ is invertible (see Lemma A.5 appendix).

Since the bird will not change its length we set the first coordinate of Φ to x_1, that is $\Phi_1(x) = x_1$. Finally, since we want to compare different shapes of birds with the same size we demand that

$$\int_{\Phi(B)} dx = \int_B dx.$$

The definition of our set of admissible mappings will be

Definition 1.1

$$\mathcal{A} = \{\Phi \in W^{2,\infty}(U); ||\Phi - Id||_{W^{2,\infty}(U)} \leq k, ||\Phi' - I||_2 \leq 1/2,$$

$$\Phi_1(x) = x_1, \Phi(x) = x \ if \ x \in F, \int_{\Phi(B)} dx = \int_B dx\}$$

The full problem can now be stated as

Problem. *Is there a $\Phi_0 \in \mathcal{A}$ such that*

$$E(\Phi_0) \leq E(\Phi), \ \forall \Phi \in \mathcal{A}$$

where

$$E(\Phi) = \int_{\Phi(D)} |\nabla u_\Phi|^2 \, dx$$

and

$$\left\{ \begin{array}{rl} -\nu \Delta u_\Phi + \sum_{i=1}^n u_{\Phi i} \partial_i u_\Phi + \nabla p_\Phi = 0 & in \ \Phi(D) \\ \mathrm{div} \ u_\Phi = 0 & in \ \Phi(D) \\ u_\Phi = f \circ \Phi^{-1} & on \ \partial\Phi(D) \end{array} \right.$$

with $\mathcal{A}$ defined as above.

Chapter 2

Difference of Solutions

In order to be able to show that there exists a minimum for our problem we need to establish some properties for the energy integral I. It is well known from basic courses in analysis that a function $f : \mathbb{R} \to \mathbb{R}$ has a minimum (or maximum) on a subset S if the function f is continuous on S and S is compact. Our setting here is a little bit more complicated, but we still have something similar. For general variational problems we have (due to [5], see Appendix) the following:

Problem A.1 *Is there a $\Phi_0 \in \mathcal{S}$ such that*

$$E(\Phi_0) \leq E(\Phi) \quad \forall \Phi \in \mathcal{S}$$

Theorem A.1 *Problem A.1 has a solution if and only if there exists a topology on $\mathcal{S}$ such that*

1. *For all $\alpha \in \mathbb{R}$ the sublevel sets*

$$\mathcal{S}_\alpha = \{\Phi \in \mathcal{S}; E(\Phi) \leq \alpha\}$$

are closed.

2. *There exists an $\alpha_0 \in \mathbb{R}$ such that $\mathcal{S}_{\alpha_0}$ is closed and compact.*

To be able to talk about closed and compact sets we must have something like a notion of distance. In this chapter we will try to find inequalities which says something like: *If u_Φ and u_Υ solves the Navier-Stokes equations in* $\Phi(D)$ *and* $\Upsilon(D)$ *respectively, then* $u_\Phi \circ \Phi \to u_\Upsilon \circ \Upsilon$ *if* $\Phi \to \Upsilon$. This will later be used to prove that the conditions of Theorem A.1 are fulfilled.

2.1 Problem stated with Laplace's equation

In this section we will study a scalar model problem similar to our. Instead of looking at our original problem we will see what happens if we let the governing equation be the Laplacian. The reason for this is, that Laplace's equation is similar to Stokes' equations.

The simplified problem will be:

Problem 1 *Is there a* $\Phi_0 \in \mathcal{A}$ *such that* $E(\Phi_0) \leq E(\Phi)$, $\forall \Phi \in \mathcal{A}$. *Where*

$$E(\Phi) = \int_{\Phi(D)} |\nabla u_\Phi|^2 \, dx$$

and u_Φ *is a solution to*

$$\begin{cases} -\Delta u_\Phi &= 0 & \text{in } \Phi(D) \\ u_\Phi &= f \circ \Phi^{-1} & \text{on } \Phi(\partial D) \end{cases} \tag{2.1}$$

and $\mathcal{A}$ *as in Definition 1.1. (see section 1.3.2)*

Remark. In the definition of Problem 1 above we say that u solves Laplace's equation in a classical sense, that is u is at least twice differentiable. However this will not always be necessary. For example, to prove the following H^2-bound we only need that u solves the weak formulation (see (A.1) Appendix) of the problem with $f \in H^{3/2}$. We will always assume that the solutions are as regular as we need them to be.

2.1.1 Difference between solutions in H^2-norm

We now look at what happens to the solution u if we apply a mapping Φ close to Id in $W^{2,\infty}$-norm. We let u be the solution of Laplace's equation in D, and u_Φ the solution in $\Phi(D)$. We will try to prove that if Φ tends to Id then u_Φ tends to u. We have

$$\begin{cases} -\Delta u_\Phi &= 0 & \text{in } \Phi(D) \\ u_\Phi &= f \circ \Psi & \text{on } \Phi(\partial D) \end{cases} \quad \begin{cases} -\Delta u &= 0 & \text{in } D \\ u &= f & \text{on } \partial D \end{cases}$$

We remind the reader that $\Psi = \Phi^{-1}$. Put $w = u - u_\Phi \circ \Phi$. This yields

$$\begin{cases} -\Delta w &= -\Delta(u - u_\Phi \circ \Phi) \;=\; \Delta(u_\Phi \circ \Phi) & \text{in } D \\ w &= f - f \circ \Psi \circ \Phi = \; 0 & \text{on } \partial D \end{cases}$$

Since w fulfills Poisson's equation with source term $f = \Delta(u_\Phi \circ \Phi)$ we get, due to (A.3), the bound $||w||_{H^2(D)} \leq ||\Delta(u_\Phi \circ \Phi)||_{L^2(D)}$. We now look at the derivatives of $u_\Phi \circ \Phi$. Due to the chain rule we deduce that

$$\partial_i(u_\Phi \circ \Phi) = \sum_j (((\partial_j u_\Phi) \circ \Phi) \; \partial_i \Phi_j)$$

and

$$\partial_{ii}(u_\Phi \circ \Phi) = \sum_{j,k} ((\partial_{jk} u_\Phi) \circ \Phi) \; \partial_i \Phi_j \partial_i \Phi_k + \sum_j ((\partial_j u_\Phi) \circ \Phi) \; \partial_{ii} \Phi_j$$

In matrix notation $\Delta(u_\Phi \circ \Phi)$ can be written

$$\sum_i \partial_{ii}(u_\Phi \circ \Phi) = \sum_i \big((\partial_i \Phi)^T ((D^2 u_\Phi) \circ \Phi)(\partial_i \Phi) + ((Du_\Phi) \circ \Phi)(\partial_{ii} \Phi)\big) \tag{2.2}$$

Since $\Delta u_\Phi = 0$ in $\Phi(D)$ we know that

$$\operatorname{tr}((D^2 u_\Phi) \circ \Phi) = \sum_i e_i^T ((D^2 u_\Phi) \circ \Phi) e_i = 0$$

where e_i is the vector with zeros at all entries except at position i where it is one. For the first term in (2.2) we now get

$$\sum_i (\partial_i \Phi)^T ((D^2 u_\Phi) \circ \Phi)(\partial_i \Phi) =$$

$$\sum_i \big((\partial_i \Phi - e_i)^T ((D^2 u_\Phi) \circ \Phi)(\partial_i \Phi - e_i) + e_i^T ((D^2 u_\Phi) \circ \Phi)(\partial_i \Phi) + (\partial_i \Phi)^T ((D^2 u_\Phi) \circ \Phi) e_i \big) =$$

$$\sum_i ((\partial_i \Phi - e_i)^T ((D^2 u_\Phi) \circ \Phi)(\partial_i \Phi - e_i)$$
$$+ e_i^T ((D^2 u_\Phi) \circ \Phi)(\partial_i \Phi - e_i) + (\partial_i \Phi - e_i)^T ((D^2 u_\Phi) \circ \Phi) e_i) =$$
$$\sum_i ((\partial_i (\Phi - Id))^T ((D^2 u_\Phi) \circ \Phi)(\partial_i (\Phi - Id)) +$$
$$e_i^T ((D^2 u_\Phi) \circ \Phi)(\partial_i (\Phi - Id)) + (\partial_i (\Phi - Id))^T ((D^2 u_\Phi) \circ \Phi)\ e_i)$$

In the second term in (2.2) we see that we can immediately replace $\partial_{ii}\Phi$ by $\partial_{ii}(\Phi - Id)$ since $\partial_{ii} Id = 0$. Hence,

$$||\Delta(u_\Phi \circ \Phi)||^2_{L^2(D)} \leq$$

$$c \sum_i \Big(\int_D |(\partial_i (\Phi - Id))^T ((D^2 u_\Phi) \circ \Phi)(\partial_i (\Phi - Id))|^2 \, dx +$$
$$\int_D |e_i^T ((D^2 u_\Phi) \circ \Phi)(\partial_i (\Phi - Id))|^2 \, dx +$$
$$\int_D |(\partial_i (\Phi - Id))^T ((D^2 u_\Phi) \circ \Phi) e_i|^2 \, dx + \int_D |((D u_\Phi) \circ \Phi)(\partial_{ii} (\Phi - Id))|^2 \, dx \Big) \leq$$

$$c \sum_i \Bigg(||\partial_i (\Phi - Id)||^4_{L^\infty(U)} \sum_{j,k} \int_D |(\partial_{kj} u_\Phi) \circ \Phi|^2 \, dx +$$
$$||\partial_i (\Phi - Id)||^2_{L^\infty(U)} \sum_j \int_D |(\partial_{ij} u_\Phi) \circ \Phi|^2 \, dx +$$
$$||\partial_i (\Phi - Id)||^2_{L^\infty(U)} \sum_j \int_D |(\partial_{ji} u_\Phi) \circ \Phi|^2 \, dx +$$
$$||\partial_{ii} (\Phi - Id)||^2_{L^\infty(D)} \sum_j \int_D |(\partial_j u_\Phi) \circ \Phi|^2 \, dx \Bigg) \leq$$

$$c || \det(\Psi') ||_{L^\infty(\Phi(D))} ||\Phi - Id||^2_{W^{2,\infty}(U)} \Big((||\Phi - Id||^2_{W^{2,\infty}(U)} + 1) ||u_\Phi||^2_{H^2(\Phi(D))} + ||u_\Phi||^2_{H^1(\Phi(D))} \Big)$$

We have, due to (A.4), that u_Φ is bounded by its boundary value $f \circ \Psi$. Due to Corollary A.1 we have that $||f \circ \Psi||_{H^{3/2}(\Phi(D))}$ is bounded by $||f||_{H^{3/2}(D)}$. In fact Corollary A.1 is only proven for H^s where s is positive integer, however this is true for all positive s due to ??.

This gives us our first theorem.

Theorem 2.1 *If Φ is a 2-diffeomorphism with $||\Psi||_{W^{2,\infty}(U)} \leq c_\Psi$ and u_Φ and u solves Laplace's equation in $\Phi(D)$ and D respectively then $||u - u_\Phi \circ \Phi||^2_{H^2(D)} \leq$*

$$c|| \det \Psi'||_{L^\infty(U)}||\Phi - Id||^2_{W^{2,\infty}(U)}(1 + ||\Phi - Id||^2_{W^{2,\infty}(U)})||f||^2_{H^{3/2}(\partial D)}$$

where c depends on c_Ψ, the dimension and the constant in Poincaré's lemma.

2.1.2 Difference between solutions in H^1-norm

Unfortunately the bound in H^2-norm will not be good enough for our needs. The reason for this is that, as we will see, $W^{2,\infty}$ is not compact in any reasonable topology suitable for the problem at hand. This means that we are unable to say that the sublevel-sets $\mathcal{A}_\alpha$ (see section 3.1.2) are compact. However $W^{2,\infty}(U)$ is compactly imbedded in $W^{1,\infty}$ so a bound in H^1 will help us.

Here we use the same notation as in section 2.1.1. If u_Φ and u are solutions to Laplace's equation in $\Phi(D)$ and D we have in the weak formulation (A.1) (see appendix) that

$$\int_D \nabla u \cdot \nabla \chi \, dx = 0 \quad \forall \chi \in H^1_0(D) \tag{2.3}$$

and

$$\int_{\Phi(D)} \nabla u_\Phi \cdot \nabla \xi \, dx = 0 \quad \forall \xi \in H^1_0(\Phi(D)) \tag{2.4}$$

Again we put $w = u - u_\Phi \circ \Phi$ and due to (2.3) get

$$\int_D \nabla w \cdot \nabla \chi \, dx = \int_D \nabla(-u_\Phi \circ \Phi) \cdot \nabla \chi \, dx \tag{2.5}$$

Differentiating yields

$$\int_D \nabla w \cdot \nabla \chi \, dx = -\int_D \sum_k \left(\sum_i ((\partial_i u_\Phi) \circ \Phi)\partial_k \Phi_i) \, \partial_k \chi \right) dx =$$

$$-\int_D \left(\sum_{k,i} ((\partial_i u_\Phi) \circ \Phi)\partial_k(\Phi_i - x_i)\partial_k \chi + \sum_k ((\partial_k u_\Phi) \circ \Phi) \, \partial_k \chi \right) dx =$$

$$-\int_D ((D(\Phi - Id))((\nabla u_\Phi) \circ \Phi)) \cdot \nabla \chi \, dx - \int_D ((\nabla u_\Phi) \circ \Phi) \cdot \nabla \chi \, dx$$

We see that since $w \in H^1_0$ we can put $\chi = w$. This gives us

$$||\nabla w||^2_{L^2(D)} = \int_D |\nabla w|^2 \, dx =$$

$$-\int_D ((D(\Phi - Id))((\nabla u_\Phi)\circ\Phi))\cdot\nabla w\,dx - \int_D ((\nabla u_\Phi)\circ\Phi)\cdot\nabla w\,dx =$$

$$-\int_D ((D(\Phi - Id))((\nabla u_\Phi)\circ\Phi))\cdot\nabla w\,dx -$$

$$\int_D (|\det\Phi'| - 1)((\nabla u_\Phi)\circ\Phi)\cdot\nabla w\,dx - \int_D |\det\Phi'|((\nabla u_\Phi)\circ\Phi)\cdot\nabla w\,dx \quad (2.6)$$

The second integral in (2.6) will tend to zero because $\det\Phi'$ tends to 1. For the third term we have, due to (2.4),

$$\int_D |\det\Phi'|((\nabla u_\Phi)\circ\Phi)\cdot\nabla w\,dx = \int_{\Phi(D)} \nabla u_\Phi\cdot(((\nabla w)\circ\Psi) - \nabla(w\circ\Psi))\,dx$$

$$= \int_{\Phi(D)} \nabla u_\Phi\cdot((D(Id-\Psi))((\nabla w)\circ\Psi))\,dx$$

Equation (2.6) now becomes

$$||\nabla w||^2_{L^2(D)} = -\int_D ((D(\Phi - Id))((\nabla u_\Phi)\circ\Phi))\cdot\nabla w\,dx$$

$$-\int_D (|\det\Phi'|-1)((\nabla u_\Phi)\circ\Phi)\cdot\nabla w\,dx - \int_{\Phi(D)} \nabla u_\Phi\cdot((D(Id-\Psi))((\nabla w)\circ\Psi))\,dx$$

Using Cauchy-Schwarz inequality on (2.6) we now obtain

$$||\nabla w||^2_{L^2(D)} \le ||(D(\Phi - Id))((\nabla u_\Phi)\circ\Phi)||_{L^2(D)}||\nabla w||_{L^2(D)} +$$

$$||(|\det\Phi'| - 1)((\nabla u_\Phi)\circ\Phi)||_{L^2(D)}||\nabla w||_{L^2(D)} +$$

$$||\nabla u_\Phi||_{L^2(\Phi(D))}||(D(\Psi - Id))((\nabla w)\circ\Psi)||_{L^2(\Phi(D))} \le$$

$$c||\Phi - Id||_{W^{1,\infty}(U)}||(\nabla u_\Phi)\circ\Phi||_{L^2(D)}||\nabla w||_{L^2(D)} +$$

$$c||\det\Phi' - 1||_{W^{1,\infty}(U)}||(\nabla u_\Phi)\circ\Phi||_{L^2(D)}||\nabla w||_{L^2(D)} +$$

$$c||\Psi - Id||_{W^{1,\infty}(U)}||\nabla u_\Phi||_{L^2(\Phi(D))}||(\nabla w)\circ\Psi||_{L^2(\Phi(D))} \le$$

$$c||\Phi - Id||_{W^{1,\infty}(U)}||\det\Psi'||_{L^\infty(U)}||f\circ\Psi||_{H^{1/2}(\Phi(D))}||\nabla w||_{L^2(D)} +$$

$$c||\det\Phi' - 1||_{L^2(U)}||\det\Psi'||_{L^\infty(U)}||f\circ\Psi||_{H^{1/2}(\Phi(D))}||\nabla w||_{L^2(D)} +$$

$$c||\Psi - Id||_{W^{1,\infty}(U)}||f\circ\Psi||_{H^{1/2}(\Phi(D))}||(\nabla w)\circ\Psi||_{L^2(\Phi(D))}$$

If Φ is in a 1-diffeomorphism then, by corollary A.1, lemma A.7 and lemma A.8 (see appendix) then

$$||f\circ\Psi||_{H^{1/2}(\Phi(D))} \le c||f||_{H^{1/2}(D)}$$

$$||\det\Phi' - 1||_{L^2(U)} \le c||\Phi - Id||_{W^{1,\infty}(U)}$$

$$||\Psi - Id||_{W^{1,\infty}(U)} \le c||\Phi - Id||_{W^{1,\infty}(U)}$$

$$||\nabla w\circ\Psi||_{L^2(\Phi(D))} \le c||\nabla w||_{L^2(D)}$$

Note that these constants all depend on the $W^{1,\infty}(D)$-norm of Φ and Ψ. However, if Φ is in $\mathcal{A}$ they become independent of Φ and Ψ because of the bound $||\Phi - Id||_{W^{2,\infty}(U)} \leq k$ and $||\Phi' - I||_2 \leq 1/2$. Due to Poincaré's inequality we also have $||w||_{L^2(D)} \leq c||\nabla w||_{L^2(D)}$. Dividing by $||\nabla w||_{L^2(D)}$ and using Poincaré's inequality gives us our H^1 bound.

Theorem 2.2 *If Φ is a 1-diffeomorphism with $||\Phi||_{W^{1,\infty}(U)} \leq c_\Phi$, $||\Psi||_{W^{1,\infty}(U)} \leq c_\Psi$ and if u_Φ and u solves Laplace's equation in $\Phi(D)$ and D respectively, then*

$$||u - u_\Phi \circ \Phi||_{H^1(D)} \leq c||\Phi - Id||_{W^{1,\infty}(U)}||f||_{H^{1/2}(D)}(||\det \Psi'||_{L^\infty(U)} + 1)$$

where c depends on c_Φ, c_Ψ, the dimension and the constant in Poincaré's lemma.

Remark We see that since the Laplacian is linear we can differentiate and obtain bounds for the derivatives. We will then get something like

$$||u - u_\Phi \circ \Phi||_{H^s(D)} \leq g(||\det \Psi'||_{L^\infty(U)})h(||\Phi - Id||_{W^{s,\infty}(U)})||f||_{H^{s-1/2}(\partial D)} \quad (2.7)$$

where $g(x)$ is bounded, $h(x) \to 0$ when $x \to 0$, if s is a positive integer. We see that by lemma A.10 (appendix), theorems 2.1, 2.2 we can get the the following corollary.

Corollary 2.1 *If Φ is an s-diffeomorphism with $||\Phi||_{W^{s,\infty}(U)} \leq c_\Phi$, $||\Phi^{-1}||_{W^{s,\infty}(U)} \leq c_{\Phi^{-1}}$, and if u_Φ and u solves Laplace's equation in $\Phi(D)$ and D respectively and $\frac{1}{\alpha} \leq ||\det \Phi'||_{L^\infty(U)} \leq \alpha$ then*

$$||u - u_\Phi \circ \Phi||_{H^s(D)} \leq c||\Phi - Id||_{W^{s,\infty}(U)}||f||_{H^{s-1/2}(\partial D)}$$

where $s = 1$ or 2 and c depends on c_Φ, $c_{\Phi^{-1}}$, α, U and the dimension.

Together with Corollary A.1, Corollary 2.1 this gives us

Corollary 2.2 *If Φ and Υ are s-diffeomorphisms with $||\Phi||_{W^{s,\infty}(U)} \leq c_\Phi$, $||\Phi^{-1}||_{W^{s,\infty}(U)} \leq c_{\Phi^{-1}}, ||\Upsilon||_{W^{s,\infty}(U)} \leq c_\Upsilon$, $||\Upsilon^{-1}||_{W^{s,\infty}(U)} \leq c_{\Upsilon^{-1}}$, and $\frac{1}{\alpha} \leq ||\det(\Upsilon \circ \Phi^{-1})'||_{L^\infty(U)} \leq \alpha$ then*

$$||u_\Phi \circ \Phi - u_\Upsilon \circ \Upsilon||_{H^s(D)} \leq c||\Phi - \Upsilon||_{W^{s,\infty}(U)}||f||_{H^{s-1/2}(D)}$$

where $s = 1$ or 2 and c depends on c_Φ, $c_{\Phi^{-1}}$, c_Υ, $c_{\Upsilon^{-1}}$, α, U and the dimension.

2.1.3 A bound on $|E(\Upsilon) - E(\Phi)|$

We will now try to find a bound on the difference of $|E(\Upsilon) - E(\Phi)|$ where $\Phi, \Upsilon \in \mathcal{A}$, and I is our objective function $E(\Phi) = \int_{\Phi(D)} |\nabla u_\Phi|^2 \, dx$ where u_Φ solves some equations in $\Phi(D)$. As we shall see we will need this to be able to say that the sublevel-sets $\mathcal{A}_\alpha$ are compact. We have

$$|E(Id) - E(\Phi)| = \left| \int_D |\nabla u|^2 \, dx - \int_{\Phi(D)} |\nabla u_\Phi|^2 \, dx \right| =$$

$$\left|\int_D |\nabla u|^2\,dx - \int_D |(\nabla u_\Phi)\circ\Phi|^2|\det\Phi'|\,dx\right| \le$$

$$\left|\int_D |\nabla u|^2 - |(\nabla u_\Phi)\circ\Phi|^2\,dx\right| + ||\det\Phi' - 1||_{L^\infty(U)}\int_D |(\nabla u_\Phi)\circ\Phi|^2\,dx \qquad (2.8)$$

This gives us two terms which both will be bounded by f and $\Phi - Id$. For the second term in (2.8) we have due to lemma A.9

$$||\det\Phi' - 1||_{L^\infty(U)}\int_D |\nabla u_\Phi|^2\,dx \le c||\Phi - Id||_{W^{1\infty}(U)}||(\nabla u_\Phi)\circ\Phi||^2_{L^2(D)} \le$$

$$c||\Phi - Id||_{W^{1\infty}(U)}||\det\Psi'||_{L^\infty(U)}||f\circ\Psi||^2_{H^{1/2}(\Phi(\partial D))} \le$$

$$c||\det\Psi'||_{L^\infty(U)}||\Phi - Id||_{W^{1\infty}(U)}||f||^2_{H^{1/2}(\partial D)}$$

We note that the first term in (2.8) is in fact

$$\left|||\nabla u||^2_{L^2(D)} - ||(\nabla u_\Phi)\circ\Phi||^2_{L^2(D)}\right| \le$$

$$(||\nabla u||_{L^2(D)} + ||(\nabla u_\Phi)\circ\Phi||_{L^2(D)})\left|||\nabla u||_{L^2(D)} - ||(\nabla u_\Phi)\circ\Phi||_{L^2(D)}\right|$$

we have that

$$||\nabla u||_{L^2(D)} + ||(\nabla u_\Phi)\circ\Phi||_{L^2(D)} \le c(1 + ||\det\Psi'||^{1/2}_{L^\infty(U)})||f||_{H^{1/2}(\partial D)}$$

we also have due to the reversed triangle inequality that

$$\left|||\nabla u||_{L^2(D)} - ||(\nabla u_\Phi)\circ\Phi||_{L^2(D)}\right| \le ||\nabla u - (\nabla u_\Phi)\circ\Phi||_{L^2(D)} =$$

$$||(\nabla u - \nabla(u_\Phi\circ\Phi)) + (\nabla(u_\Phi\circ\Phi) - (\nabla u_\Phi)\circ\Phi)||_{L^2(D)} \le$$

$$||\nabla u - \nabla(u_\Phi\circ\Phi)||_{L^2(D)} + ||(D(\Phi - Id))((\nabla u_\Phi)\circ\Phi)||_{L^2(D)} \le$$

$$c||\det\Psi'||^{1/2}_{L^\infty(U)}||\Phi - Id||_{W^{1,\infty}(U)}||f||_{H^{1/2}(\partial D)}$$

Note that some of the constants depend on how big $||\Phi||_{W^{1,\infty}(U)}$ and $||\Psi||_{W^{1,\infty}(u)}$ is. Since we know that Φ is in $\mathcal{A}$ we have that $||\Phi||_{W^{1,\infty}(U)}$ and $||\Psi||_{W^{1,\infty}(U)}$ is bounded. Therefore we get the following theorem

Theorem 2.3 *If $\Phi \in \mathcal{A}$ and $\frac{1}{\alpha} \le ||\det\Phi'||_{L^\infty(U)} \le \alpha$ then*

$$|E(Id) - E(\Phi)| \le c||\Phi - Id||_{W^{1,\infty}(U)}||f||^2_{H^{1/2}(\partial D)}$$

where c depends on U, α and the dimension.

Now we note that if we replace Id by Υ we get

$$|E(\Phi) - E(\Upsilon)| \le \left|\int_{\Phi(D)} |\nabla u_\Phi|^2 dx - \int_{\Phi(D)} |\det(\Upsilon\circ\Phi^{-1})'||\nabla u_\Upsilon\circ\Upsilon\circ\Phi^{-1}|^2 dx\right| \le$$

$$||\det(\Upsilon\circ\Phi^{-1})' - 1||_{L^\infty(U)}\int_{\Phi(D)} |\nabla u_\Upsilon\circ\Upsilon\circ\Phi^{-1}|^2\,dx +$$

$$\int_{\Phi(D)} \left(|\nabla u_\Phi|^2 - |\nabla u_\Upsilon\circ\Upsilon\circ\Phi^{-1}|^2\right)\,dx$$

Since we know that Υ is also in $\mathcal{A}$ we can, in exactly the same way as we have done above, prove that

$$|E(\Phi) - E(\Upsilon)| \leq c||\Upsilon \circ \Phi^{-1} - Id||_{W^{1,\infty}(U)}||f||^2_{H^{1/2}(\partial D)} \tag{2.9}$$

And finally corollary A.1 now gives us

Corollary 2.3 *If Φ and Υ are in $\mathcal{A}$, and $\frac{1}{\alpha} \leq ||\det(\Upsilon \circ \Phi^{-1})'||_{L^\infty(U)} \leq \alpha$ then*

$$|E(\Phi) - E(\Upsilon)| \leq c||\Upsilon \circ \Phi^{-1} - Id||_{W^{1,\infty}(U)}||f||^2_{H^{1/2}(\partial D)} \leq$$

$$c||\Upsilon - \Phi||_{W^{1,\infty}(U)}||f||^2_{H^{1/2}(\partial D)}$$

where c depends on α, U and the dimension.

Remark. This means that the mapping $\Phi \mapsto E(\Phi)$ is Lipschitz continuous from $\mathcal{A}$ to $\mathbb{R}$.

2.2 Problem stated with Stokes equations

In this section we will look at what happens if we let the governing equations be the Stokes equations. We will try to find the same bounds as for the Laplacian and a lot of the calculations will be the same. Our problem will be

Problem 2 *Is there a $\Phi_0 \in \mathcal{A}$ such that $E(\Phi_0) \leq E(\Phi)$ $\forall \Phi \in \mathcal{A}$. Where*

$$E(\Phi) = \int_{\Phi(D)} |\nabla u_\Phi|^2 \, dx$$

and u_Φ is a solution to

$$\begin{cases} -\nu\Delta u_\Phi + \nabla p_\Phi &= 0 & in\ \Phi(D) \\ \text{div } u_\Phi &= 0 & on\ \Phi(D) \\ u_\Phi &= f \circ \Psi & on\ \Phi(\partial D) \end{cases}$$

with $\mathcal{A}$ defined as in Definition 1.1 (see section 1.3.2).

2.2.1 Difference of solutions in H^2-norm II

We start by looking at the difference of solutions for the Stokes equations in different domains. We have

$$\begin{cases} -\nu\Delta u_\Phi + \nabla p_\Phi &= 0 & in\ \Phi(D) \\ \text{div } u_\Phi &= 0 & in\ \Phi(D) \\ u_\Phi &= f \circ \Psi & on\ \Phi(\partial D) \end{cases} \quad \begin{cases} -\nu\Delta u + \nabla p &= 0 & in\ D \\ \text{div } u &= 0 & in\ D \\ u &= f & on\ \partial D \end{cases}$$

Put $w = u - u_\Phi \circ \Phi$, and $q = p - p_\Phi \circ \Phi$. We then get

$$\begin{cases} -\nu\Delta w + \nabla q = \nu\Delta(u_\Phi \circ \Phi) - \nabla(p_\Phi \circ \Phi) &=: R & in\ D \\ \text{div } w = -\text{div } (u_\Phi \circ \Phi) &=: g & in\ D \\ w &= 0 & on\ \partial D \end{cases}$$

According to (A.7) we then have

$$||w||_{H^2(D)} + ||q||_{H^1(D)/\mathbb{R}} \leq c_0(||R||_{L^2(D)} + ||g||_{H^1(D)})$$

First we look at the j:th component of R. We have

$$R_j = -\nu \left(\sum_{i,k,l} ((\partial_{kl} u_{\Phi j}) \circ \Phi) \partial_i \Phi_k \partial_i \Phi_l + \right.$$

$$\left. \sum_{i,k} ((\partial_k u_{\Phi j}) \circ \Phi) \partial_{ii} \Phi_k \right) + \sum_k ((\partial_k p_\Phi) \circ \Phi) \partial_j \Phi_k =$$

$$-\nu \left(\sum_i (\partial_i \Phi)^T ((D^2 u_{\Phi j}) \circ \Phi)(\partial_i \Phi) + \sum_i ((D u_{\Phi j}) \circ \Phi)(\partial_{ii} \Phi) \right) +$$

$$((D p_\Phi) \circ \Phi)(\partial_j \Phi) \tag{2.10}$$

Since u_Φ and p_Φ solves Stokes equations we know that

$$-\nu \Delta u_{\Phi j} + \partial_j p_\Phi = 0 \Leftrightarrow -\nu \sum_i e_i^T ((D^2 u_{\Phi j}) \circ \Phi) e_i + ((D p_\Phi) \circ \Phi) e_j = 0 \tag{2.11}$$

This gives us that the first term in (2.10) can be written as

$$-\nu \sum_i (\partial_i \Phi)^T ((D^2 u_{\Phi j}) \circ \Phi)(\partial_i \Phi) = -\nu \sum_i \big((\partial_i \Phi - e_i)^T ((D^2 u_{\Phi j}) \circ \Phi)(\partial_i \Phi - e_i) +$$

$$e_i^T ((D^2 u_{\Phi j}) \circ \Phi)(\partial_i \Phi - e_i) + e_i^T ((D^2 u_{\Phi j}) \circ \Phi)(\partial_i \Phi - e_i) \big) - ((D p_\Phi) \circ \Phi) e_j \tag{2.12}$$

In the second term of (2.10) we can immediately insert $\Phi - Id$ since $\partial_{ii} Id = 0$. This gives us

$$||R_j||^2_{L^2(D)} \leq$$

$$c \left(\sum_i \left(\int_D |(\partial_i (\Phi - Id))^T ((D^2 u_{\Phi j}) \circ \Phi)(\partial_i (\Phi - Id))|^2 \, dx \; + \right. \right.$$

$$\int_D |(\partial_i (\Phi - Id))^T ((D^2 u_{\Phi j}) \circ \Phi) e_i|^2 \, dx +$$

$$\int_D e_i^T ((D^2 u_{\Phi j}) \circ \Phi)(\partial_i (\Phi - Id)) \, dx +$$

$$\left. \left. \int_D |((D u_{\Phi j}) \circ \Phi)(\partial_{ii} (\Phi - Id))|^2 \, dx \right) + ((D p_\Phi) \circ \Phi)(\partial_j (\Phi - Id)) \right) \tag{2.13}$$

Since both p_Φ and u_Φ are bounded by $f \circ \Psi$ we get

$$||R||^2_{L^2(D)} \leq c || \det \Psi' ||_{L^\infty(\Phi(D))} ||\Phi - Id||^2_{W^{2,\infty}(U)} (||\Phi - Id||^2_{W^{2,\infty}(U)} +$$

$$1) ||f \circ \Psi||^2_{H^{3/2}(\Phi(\partial D))} \tag{2.14}$$

We also have to look for a H^1 bound of the divergence term g. Since v solves Stokes equations we have that

$$\sum_j ((Du_{\Phi j}) \circ \Phi) e_j = (\text{div } u_\Phi) \circ \Phi = 0 \tag{2.15}$$

For g we have

$$g = \sum_j \partial_j (u_{\Phi j} \circ \Phi) = \sum_{j,k} ((\partial_k u_{\Phi j}) \circ \Phi) \partial_j \Phi_k = \sum_j ((Du_{\Phi j}) \circ \Phi) \partial_j \Phi =$$

$$\sum_j ((Du_{\Phi j}) \circ \Phi)(\partial_j \Phi - e_j) + \sum_i ((Du_{\Phi j}) \circ \Phi) e_j$$

$$= \sum_j ((Du_{\Phi j}) \circ \Phi)(\partial_j \Phi - e_j) \tag{2.16}$$

In the same way as before, this yields

$$||\text{div } w||^2_{L^2(D)} \leq c|| \det \Psi'||_{L^\infty(\Phi(D))} ||\Phi - Id||^2_{W^{1,\infty}(U)} ||f \circ \Psi||^2_{H^{1/2}(\Phi(\partial D))} \tag{2.17}$$

For the derivatives we get

$$\partial_i g = \partial_i \sum_j ((Du_{\Phi j}) \circ \Phi)(\partial_j \Phi - e_j) = \sum_j (\partial_j \Phi - e_j)^T ((D^2 u_{\Phi j}) \circ \Phi)(\partial_i \Phi) +$$

$$\sum_j ((Du_{\Phi j}) \circ \Phi)(\partial_{ij}(\Phi - Id)) =$$

$$\sum_j ((\partial_j \Phi - e_j)^T ((D^2 u_{\Phi j}) \circ \Phi)(\partial_i \Phi - e_i) +$$

$$(\partial_j \Phi - e_j)^T ((D^2 u_{\Phi j}) \circ \Phi) e_j + ((Du_{\Phi j}) \circ \Phi)(\partial_{ij}(\Phi - Id))) \tag{2.18}$$

and we can deduce the following bound on the derivatives

$$||\partial_i g||^2_{L^2(D)} \leq c|| \det \Psi'||_{L^\infty(\Phi(D))} ||\Phi - Id||^2_{W^{2,\infty}(U)} (||\Phi - Id||^2_{W^{2,\infty}(U)} +$$

$$1) ||f \circ \Psi||^2_{H^{3/2}(\Phi(\partial D))} \tag{2.19}$$

Now we can add all terms together and get our H^2 bound on the difference of the solutions

Theorem 2.4 *If Φ is a 2-diffeomorphism with $||\Psi||_{W^{2,\infty}(U)} \leq c_\Psi$ and u_Φ, p_Φ and u, p solves Stokes equations in $\Phi(D)$ and D respectively then*

$$||u - u_\Phi \circ \Phi||^2_{H^2(D)} + ||p - p_\Phi \circ \Phi||^2_{H^1(D)/\mathbb{R}} \leq$$

$$c|| \det \Psi'||_{L^\infty(U)} ||\Phi - Id||^2_{W^{2,\infty}(U)} (1 + ||\Phi - Id||^2_{W^{2,\infty}(U)}) ||f||^2_{H^{3/2}(\partial D)}$$

where c depends on c_Ψ, U and the dimension.

2.2.2 Difference of solutions in H^1-norm II

We will now try to find a H^1-bound for the Stokes equations. We use the same notation as in the previous section. Since the regularity result (A.7) is valid for $m = -1$ we have

$$||w||_{H^1(D)} + ||q||_{L^2(D)/\mathbb{R}} \leq c_0(||R||_{H^{-1}(D)} + ||g||_{L^2(D)}) \tag{2.20}$$

By (2.17) we already have the $||g||_{L^2(D)}$ bound. For $||R||_{H^{-1}(D)}$ we have

$$||R||_{H^{-1}(D)} = \sup_{||\chi||_{H^1(D)}=1} < R, \chi > \tag{2.21}$$

where $\chi \in H_0^1(D)$.

$$< R, \chi >= \sum_i < \nu\Delta(u_{\Phi i} \circ \Phi) - \partial_i(p_\Phi \circ \Phi), \chi_i >=$$

$$-\sum_i \left(\int_D \nu\nabla(u_{\Phi i} \circ \Phi) \cdot \nabla\chi_i \, dx \right) + \int_D (p_\Phi \circ \Phi) \text{div } \chi \, dx =$$

$$-\sum_i \left(\int_D \nu(D(\Phi - Id))((\nabla u_{\Phi i}) \circ \Phi) \cdot \nabla\chi_i \, dx + \int_D \nu((\nabla u) \circ \Phi) \cdot \nabla\chi_i \, dx \right) +$$

$$\int_D (p_\Phi \circ \Phi) \text{div } \chi \, dx \tag{2.22}$$

The first term in (2.22) will go to zero when Φ goes to Id. For the second and third term we have

$$-\nu \sum_i \left(\int_D ((\nabla u_{\Phi i}) \circ \Phi) \cdot \nabla\chi_i dx \right) + \int_D (p_\Phi \circ \Phi) \text{div } \chi \, dx =$$

$$-\nu \sum_i \left(\int_D (1 - |\det \Phi'|)((\nabla u_{\Phi i}) \circ \Phi) \cdot \nabla\chi_i \, dx \right) - \nu \sum_i \left(\int_D |\det \Phi'|((\nabla u_{\Phi i}) \circ \Phi) \cdot \nabla\chi_i \, dx \right) +$$

$$\int_D (1 - |\det \Phi'|)(p_\Phi \circ \Phi) \text{div } \chi \, dx + \int_D |\det \Phi'|(p_\Phi \circ \Phi) \text{div } \chi \, dx \tag{2.23}$$

The $(1 - |\det \Phi'|)$ terms will be bounded by $||\Phi - Id||_{W^{1,\infty}(U)}$ because of lemma A.9. For the second and fourth term in (2.23) we get

$$-\nu \sum_i \left(\int_D |\det \Phi'|((\nabla u_{\Phi i}) \circ \Phi) \cdot \nabla\chi_i \, dx \right) + \int_D |\det \Phi'|(p_\Phi \circ \Phi) \text{div } \chi \, dx =$$

$$-\nu \sum_i \left(\int_{\Phi(D)} \nabla u_{\Phi i} \cdot ((\nabla\chi_i) \circ \Psi) \, dx \right) + \int_{\Phi(D)} p_\Phi((\text{div } \chi) \circ \Psi) \, dx =$$

$$-\nu \sum_i \left(\int_{\Phi(D)} \nabla u_{\Phi i} \cdot ((\nabla\chi_i) \circ \Psi - \nabla(\chi_i \circ \Psi)) \, dx \right) - \nu \sum_i \left(\int_{\Phi(D)} \nabla u_{\Phi i} \cdot \nabla(\chi_i \circ \Psi) \, dx \right) +$$

$$\int_{\Phi(D)} p_\Phi(((\text{div } \chi) \circ \Psi) - \text{div } (\chi \circ \Psi)) \, dx + \int_{\Phi(D)} p_\Phi \text{div } (\chi \circ \Psi) \, dx \tag{2.24}$$

Since u_Φ and p_Φ solves Stokes equations and $\chi \circ \Psi \in H_0^1(\Phi(D))$ we get that the second and fourth term in (2.24) cancels. We also have that

$$\nu \sum_i \left(\int_{\Phi(D)} \nabla u_{\Phi i} \cdot ((\nabla \chi_i) \circ \Psi - \nabla(\chi_i \circ \Psi))\, dx \right) =$$

$$\nu \sum_i \left(\int_{\Phi(D)} \nabla u_{\Phi_i} \cdot (D(Id - \Psi))((\nabla \chi_i) \circ \Psi)\, dx \right) \tag{2.25}$$

For the third term in (2.24) we have

$$(\text{div } \chi) \circ \Psi - \text{div } (\chi \circ \Psi) =$$

$$\sum_i ((\partial_i \chi_i) \circ \Psi) - \sum_i (\partial_i (\chi_i) \circ \Psi) =$$

$$\sum_i ((\partial_i \chi_i) \circ \Psi) - \sum_{i,k} ((\partial_k \chi_i) \circ \Phi)(\partial_i \Psi_k) =$$

$$-\sum_{i,k} ((\partial_k \chi_i) \circ \Psi)(\partial_i (\Psi_k - x_k)) = \sum_i (\partial_i (Id - \Psi))^T ((\nabla \chi_i) \circ \Psi) \tag{2.26}$$

Combining (2.22)-(2.26) gives the expression

$$(R, \chi) = -\sum_i \left(\int_D \nu (D(\Phi - Id))((\nabla u_{\Phi i}) \circ \Phi) \cdot \nabla \chi_i \, dx \right) -$$

$$\nu \sum_i \left(\int_D (1 - |\det \Phi'|)((u_{\Phi i}) \circ \Phi) \cdot \nabla \chi_i dx \right) + \int_D (1 - |\det \Phi'|)(p_\Phi \circ \Phi) \text{div } \chi \, dx -$$

$$\nu \sum_i \left(\int_{\Phi(D)} \nabla u_{\Phi i} \cdot (D(Id - \Psi))((\nabla \chi_i) \circ \Psi)\, dx \right) +$$

$$\sum_i \left(\int_{\Phi(D)} p_\Phi (\partial_i (Id - \Psi))^T ((\nabla \chi_i) \circ \Psi)\, dx \right) \tag{2.27}$$

(2.27) now gives us the bound

$$||R||_{H^{-1}(D)} \leq c(||\Phi - Id||_{W^{1,\infty}(U)} ||(\nabla u_{\Phi i}) \circ \Phi||_{L^2(D)} ||\nabla \chi||_{L^2(D)} +$$
$$||1 - \det \Phi'||_{L^\infty(U)} ||(\nabla u_{\Phi i}) \circ \Phi||_{L^2(D)} ||\nabla \chi||_{L^2(D)} +$$
$$||1 - \det \Phi'||_{L^\infty(U)} ||p_\Phi \circ \Phi||_{L^2(D)} ||\text{div } \chi||_{L^2(D)} +$$
$$||\nabla u_\Phi||_{L^2(\Phi(D))} ||\Psi - Id||_{W^{1,\infty}(U)} ||(\nabla \chi) \circ \Psi||_{L^2(\Phi(D))} +$$
$$||p_\Phi||_{L^2(\Phi(D))} ||\Psi - Id||_{W^{1,\infty}(U)} ||(\nabla \chi) \circ \Psi||_{L^2(\Phi(D))} \tag{2.28}$$

For the terms involved we have the following bounds

$$||(\nabla u_{\Phi i}) \circ \Phi||_{L^2(D)} \leq c ||\det \Psi'||_{L^\infty(U)} ||f||_{H^{1/2}(\partial D)}$$

$$||1 - \det \Phi'||_{L^\infty(U)} \leq c ||\Phi - Id||_{W^{1,\infty}(U)}$$

$$||p_\Phi \circ \Phi||_{L^2(D)} \leq c ||\det \Psi'||_{L^\infty(U)} ||f||_{H^{1/2}(\partial D)}$$

$$||\nabla u_\Phi||_{L^2(\Phi(D))} \leq c||f||_{H^{1/2}(\partial D)}$$
$$||p_\Phi||_{L^2(\Phi(D))} \leq c||f||_{H^{1/2}(\partial D)}$$
$$||\Psi - Id||_{W^{1,\infty}(U)} \leq c||\Phi - Id||_{W^{1,\infty}(U)}$$
$$||(\nabla\chi) \circ \Psi||_{L^2(\Phi(D))} \leq c||\chi||_{H^1(D)}$$
$$||\chi||_{H^1(D)} = 1$$

where the constants depend on $||\Phi||_{W^{1,\infty}(U)}$, $||\Psi||_{W^{1,\infty}(U)}$, U and the dimension. This together with (2.17) this gives us our Lipschitz bound in H^1-norm.

Theorem 2.5 *If Φ is an 1-diffeomorphism with $||\Phi||_{W^{1,\infty}(U)} \leq c_\Phi$, $||\Psi||_{W^{1,\infty}(U)} \leq c_\Psi$ and u_Φ and u solves Stokes equations in $\Phi(D)$ and D respectively, $\frac{1}{\alpha} \leq || \det \Phi'||_{L^\infty(U)} \leq \alpha$ then*

$$||u - u_\Phi \circ \Phi||_{H^1(D)} + ||p - p_\Phi \circ \Phi||_{L^2(D)/\mathbb{R}} \leq c||\Phi - Id||_{W^{1,\infty}(U)}||f||_{H^{1/2}(\partial D)}$$

where c depends on c_Φ, c_Ψ, α, U and the dimension.

In exactly the same way as we did for the Laplacian we can now find the same bound for $|E(\Phi) - E(\Upsilon)|$.

Corollary 2.4 *If Φ and Υ are in $\mathcal{A}$, and $\frac{1}{\alpha} \leq || \det(\Upsilon \circ \Phi^{-1})'||_{L^\infty(U)} \leq \alpha$ then*

$$|E(\Phi) - E(\Upsilon)| \leq c||\Upsilon \circ \Phi^{-1} - Id||_{W^{1,\infty}(U)}||f||^2_{H^{1/2}(\partial D)} \leq$$
$$c||\Upsilon - \Phi||_{W^{1,\infty}(U)}||f||^2_{H^{1/2}(\partial D)}$$

where c depends on α, U and the dimension.

2.3 Problem stated with Navier-Stokes equations

We will now look at our real problem.

Problem 3 *Is there a $\Phi_0 \in \mathcal{A}$ such that $E(\Phi_0) \leq E(\Phi)$ $\forall \Phi \in \mathcal{A}$. Where*

$$E(\Phi) = \int_{\Phi(D)} |\nabla u_\Phi|^2 \, dx$$

and u_Φ is a solution to

$$\begin{cases} -\nu \Delta u_\Phi + \sum_{i=1}^n u_{\Phi i} \partial_i u_\Phi + \nabla p_\Phi &= 0 \quad \text{in } \Phi(D) \\ \text{div } u_\Phi &= 0 \quad \text{in } \Phi(D) \\ u_\Phi &= f \circ \Psi \quad \text{on } \Phi(\partial D) \end{cases}$$

with $\mathcal{A}$ as in Definition 1.1 (see section 1.3.2).

2.3.1 Difference of solutions in H^1-norm III

In the same way as before we will look at the difference $w = u - u_\Phi \circ \Phi$. We have

$$\begin{cases} -\nu\Delta u_\Phi + \sum_{i=1}^n u_{\Phi i}\partial_i u_\Phi + \nabla p_\Phi &= 0 \quad in\ \Phi(D) \\ \text{div } u_\Phi &= 0 \quad in\ \Phi(D) \\ u_\Phi &= f\circ\Psi \quad on\ \Phi(\partial D) \end{cases}$$

$$\begin{cases} -\nu\Delta u + \sum_{i=1}^n u_i\partial_i u + \nabla p &= 0 \quad in\ D \\ \text{div } u &= 0 \quad in\ D \\ u &= f \quad on\ \partial D \end{cases}$$

The nonlinear term will, as one might expect, cause problems. In order to avoid these, we will have to assume that the Reynolds number is sufficiently small. In the weak formulation we have that

$$\nu \sum_i \int_{\Phi(D)} \nabla u_{\Phi i}\cdot\nabla\xi_i\, dx =$$

$$\int_{\Phi(D)} \left(\sum_{i,j} u_{\Phi i}\partial_i u_{\Phi j}\xi_j + \sum_i \partial_i p_\Phi \xi_i \right) dx \quad \forall\xi\in H_0^1(\Phi(D)) \tag{2.29}$$

and

$$\nu \sum_i \int_D \nabla u\cdot\nabla\chi_i\, dx =$$

$$\int_D \left(\sum_{i,j} u_i\partial_i u_j\chi_j + \sum_i \partial_i p_\Phi \chi_i \right) dx \quad \forall\chi\in H_0^1(D) \tag{2.30}$$

Again we put $w = u - u_\Phi\circ\Phi$ and get

$$\nu \int_D \sum_i \nabla w_i\cdot\nabla\chi_i\, dx =$$

$$\nu \int_D \sum_i ((\nabla u_i - (\nabla u_{\Phi i})\circ\Phi) - ((D(\Phi - Id))((\nabla u_{\Phi i})\circ\Phi))\cdot\nabla\chi_i\, dx \tag{2.31}$$

$\forall\chi\in H_0^1(D)$. Since u_Φ is the solution of Navier-Stokes equations in $\Phi(D)$, $\nabla u_{\Phi i}$ is bounded. We see that the the term $D(\Phi - Id)(\nabla u_{\Phi i})\circ\Phi$ will then tend to zero. If we apply (2.30) to the right hand side in (2.31) and put $w = \chi$ we get

$$\nu \int_D \sum_i |\nabla w_i|^2\, dx =$$

$$\int_D \sum_{i,j} u_i\partial_i u_j w_j\, dx + \int_D p \text{ div } w\, dx$$

$$-\nu \int_{\Phi(D)} \sum_i \nabla u_{\Phi i}\cdot((\nabla w_i)\circ\Psi)\, dx \tag{2.32}$$

The second term in (2.32) will tend to zero since the divergence of w will go to zero and p is bounded. For the last term in (2.32) we have

$$-\nu \int_{\Phi(D)} \sum_i \nabla u_{\Phi i}\cdot((\nabla w_i)\circ\Psi)\, dx =$$

$$-\nu\int_{\Phi(D)}\sum_i \nabla u_{\Phi i}\cdot((\nabla w_i)\circ\Psi-\nabla(w_i\circ\Psi))\,dx-\nu\int_{\Phi(D)}\sum_i \nabla u_{\Phi i}\cdot\nabla(w_i\circ\Psi)\,dx=$$

$$\nu\int_{\Phi(D)}\sum_i \nabla u_{\Phi i}\cdot((D(\Phi-Id))((\nabla w_i)\circ\Psi))\,dx-\nu\int_{\Phi(D)}\sum_i \nabla u_{\Phi i}\cdot\nabla(w_i\circ\Psi)\,dx=$$

$$-\nu\int_{\Phi(D)}\sum_i \nabla u_{\Phi i}\cdot\nabla(w_i\circ\Psi)\,dx+h(\Phi-Id) \tag{2.33}$$

Here $h(\Phi-Id)$ is a function that tends to zero as $||\Phi-Id||_{W^{1,\infty}(U)}$ tends to zero. In order to make the calculations a bit less messy we will from now on write all terms that tends to zero as $h(\Phi-Id)$ where h is a function that tends to zero as $||\Phi-Id||_{W^{1,\infty}(U)}$ tends to zero. Note that we will keep the same symbol h even if it changes from one line to the next. We now apply (2.29) to (2.33) and get

$$-\nu\int_{\Phi(D)}\sum_i \nabla u_{\Phi i}\cdot((\nabla w_i)\circ\Psi)\,dx=$$

$$-\int_{\Phi(D)}\sum_{i,j} u_{\Phi j}\partial_j u_{\Phi i}(w_i\circ\Psi)\,dx-\int_{\Phi(D)} p_\Phi \text{ div } (w\circ\Psi)\,dx=$$

$$-\int_D|\det\Phi'|\sum_{i,j}(u_{\Phi j}\circ\Phi)((\partial_j u_{\Phi i})\circ\Phi)w_i\,dx+h(\Phi-Id) \tag{2.34}$$

For the first term in (2.34) we have

$$-\int_D|\det\Phi'|\sum_{i,j}(u_{\Phi j}\circ\Phi)((\partial_j u_{\Phi i})\circ\Phi)w_i\,dx=$$

$$-\int_D|\det\Phi'-1|\sum_{i,j}(u_{\Phi j}\circ\Phi)((\partial_j u_{\Phi i})\circ\Phi)w_i\,dx-\int_D\sum_{i,j}(u_{\Phi j}\circ\Phi)((\partial_j u_{\Phi i})\circ\Phi)w_i\,dx=$$

$$-\int_D\sum_{i,j}(u_{\Phi j}\circ\Phi)(\partial_j(u_{\Phi i}\circ\Phi))w_i\,dx-$$

$$\int_D\sum_{i,j}(u_{\Phi j}\circ\Phi)((\partial_j u_{\Phi i})\circ\Phi-\partial_j(u_{\Phi i}\circ\Phi))\;w_i\,dx+h(\Phi-Id)=$$

$$=-b(u_\Phi\circ\Phi,u_\Phi\circ\Phi,w)+h(\Phi-Id) \tag{2.35}$$

We now apply (2.35) and (2.34) to (2.32) and get

$$\nu\int_D\sum_i|\nabla w_i|^2\,dx=b(u,u,w)-b(u_\Phi\circ\Phi,u_\Phi\circ\Phi,w)+h(\Phi-Id) \tag{2.36}$$

We have due to lemma A.5 that $b(u,w,w)=0$ if $u\in H^1$ and w is in H_0^1 and div $u=0$. This gives us that the expression above is

$$b(u,u-u_\Phi\circ\Phi,w)+b(u,u_\Phi\circ\Phi,w)-b(u_\Phi\circ\Phi,u_\Phi\circ\Phi,w)+h(\Phi-Id)=$$

$$b(w,u_\Phi\circ\Phi,w)+h(\Phi-Id)\le c(n)||w||^2_{H_0^1(D)}||u_\Phi\circ\Phi||_{H_0^1(D)}+|h(\Phi-Id)|$$

$$\le c(n)||\det\Psi'||^{1/2}_{L^\infty(D)}||u_\Phi||_{H_0^1(\Phi(D))}||w||^2_{H_0^1(D)}+|h(\Phi-Id)|=$$

$$\leq c(n)||u_\Phi||_{H_0^1(\Phi(D))}||w||^2_{H_0^1(D)}+$$

$$c(n)(||\det \Psi'||^{1/2}_{L^\infty(D)} - 1)||u_\Phi||_{H_0^1(\Phi(D))}||w||^2_{H_0^1(D)} + |h(\Phi - Id)| =$$

$$c(n)||u_\Phi||_{H_0^1(\Phi(D))}||w||^2_{H_0^1(D)} + |h(\Phi - Id)| \tag{2.37}$$

Now we move all w terms to the left side and get

$$\left(\nu - c(n)||u_\Phi||_{H_0^1(\Phi(D))}\right)||w||^2_{H_0^1(D)} \leq |h(\Phi - Id)| \tag{2.38}$$

Now we see that if there is a constant k such that

$$\nu - c(n)||u_\Phi||_{H_0^1(\Phi(D))} \geq k > 0 \tag{2.39}$$

then the term on the left hand side of (2.38) will be larger than zero, and we know that

$$k||w||^2_{H_0^1(D)} \leq \left(\nu - c(n)||u_\Phi||_{H_0^1(\Phi)}\right)||w||^2_{H_0^1(D)} \leq |h(\Phi - Id)| \tag{2.40}$$

Now we need to now that the conditions (2.39) can be fulfilled. We will see that if the Reynolds number is low then they are in fact fulfilled. Due to lemma A.4 we have that there is a constant Re_0 depending on U and the dimension such that if

$$Re := \frac{||f \circ \Psi||_{H^{1/2}(\Phi(D))}}{\nu} \leq Re_0 \tag{2.41}$$

then

$$||u_\Phi||_{H^1(\Phi(D))} \leq (3k+1)c_0||f \circ \Psi||_{H^{1/2}(\Phi(D))} \tag{2.42}$$

We therefore have that if $||\Psi||_{W^{1,\infty}(U)} \leq c_\Psi$

$$\nu - c(n)||u_\Phi||_{H_0^1(\Phi(D))} \geq \nu - c(n)(3k+1)c_0c_1||f||_{H^{1/2}(D)}$$

where c_1 is a constant that depends on c_Ψ, and k and c_0 are as defined in section A.7.2 of the appendix. If $||\Psi||_{W^{1,\infty}(U)}$ is bounded then we can now choose ν large or f small such that (2.39) is fulfilled. Therefore we can conclude the following

Theorem 2.7 *There exists a constant Re_1 such that, if u_Φ and u solves the stationary Navier-Stokes equations in $\Phi(D)$ and D respectively, and $\Phi \in \mathcal{A}$, and if*

$$Re := \frac{||f||_{H^{1/2}(D)}}{\nu} \leq Re_1$$

then $||u - u_\Phi \circ \Phi||_{H^1(D)} \to 0$ if $||\Phi - Id||_{W^{1,\infty}(U)} \to 0$.

In almost the same way as for the Laplace and Stokes problems we can now prove

Corollary 2.5 *There exists a constant Re_1 such that, if Φ and Υ are in $\mathcal{A}$ and*

$$Re = \frac{||f||_{H^{1/2}(D)}}{\nu} \leq Re_1$$

then $|E(\Phi) - E(\Upsilon)| \to 0$ as $||\Phi - \Upsilon||_{W^{1,\infty}(U)} \to 0$.

Chapter 3

Existence of Solutions in $\mathcal{A}$

In this chapter we will prove existence of minimizers to our Problems 1,2 and 3 (defined in sections 2.1, 2.2 and 2.3). The main objective is to find a topology on $\mathcal{A}$ (defined in section 1.3.2) such that the conditions of theorem A.1 are fulfilled.

3.1 Compactness

We start by looking at the topology. We take a bounded sequence $\{\Phi^k\}$ in $W^{2,\infty}(U)$. The mapping

$$M : \Phi \mapsto (\Phi, \partial_1\Phi, ..., \partial_n\Phi, \partial_{11}\Phi, ..., \partial_{1n}\Phi, ..., \partial_{nn}\Phi)$$

is an isometry from $W^{2,\infty}$ to the jet-space

$$\Pi_2^* = \underbrace{L^\infty(U) \times L^\infty(U) \times ...L^\infty(U)}_{\begin{pmatrix} n \\ 2 \end{pmatrix}+n+1}$$

with norm

$$||f||_{\Pi_2^*} = ||(f_1, ..., f_n, f_{11}, ..., f_{1n}, ..., f_{nn})||_{\Pi_2^*} =$$
$$\max(||f_1||_{L^\infty(U)}, ..., ||f_n||_{L^\infty(U)}, ||f_{11}||_{L^\infty(U)}, ..., ||f_{1n}||_{L^\infty(U)}, ..., ||f_{nn}||_{L^\infty(U)})$$

Since $L^\infty(U)$ is the dual space of $L^1(U)$ we have that Π_2^* is the dual space of

$$\Pi_2 = \underbrace{L^1(U) \times L^1(U) \times ...L^1(U)}_{\begin{pmatrix} n \\ 2 \end{pmatrix}+n+1}$$

with norm

$$||f||_{\Pi_2} = ||(f_1, ..., f_n, f_{11}, ..., f_{1n}, ..., f_{nn})||_{\Pi_2} =$$
$$||f_1||_{L^1(U)} + ... + ||f_n||_{L^1(U)} + ||f_{11}||_{L^1(U)} + ... + ||f_{1n}||_{L^1(U)} + ... + ||f_{nn}||_{L^1(U)}$$

Since L^1 is separable and $M(\Phi^k)$ is bounded in Π^* norm the Banach-Alaoglu Theorem tells us that, we can find a subsequence such that

$$(\Phi^{k_j}, \partial_1\Phi^{k_j}, ..., \partial_n\Phi^{k_j}, \partial_{11}\Phi^{k_j}, ..., \partial_{1n}\Phi^{k_j}, ..., \partial_{nn}\Phi^{k_j}) \xrightarrow{*}$$

$$(v, v_1, ..., v_n, v_{11}, ..., v_{1n}, ..., v_{nn})$$

in the $\sigma(\Pi_2^*, \Pi_2)$ topology. Clearly Φ^{k_j} will now tend to v weakly-* in L^∞. However since $\mathcal{D}(U) \subset L^1(U)$ we also have convergence in the sense of distributions and therefore $\partial_{ij}\Phi_k \overset{*}{\rightharpoonup} \partial_{ij}v$. Therefore we get that the sequence Φ^{k_j} converges in the distribution sense to v, and we can thus conclude that a closed subset of $W^{2,\infty}$ is closed and compact with the $\sigma(\Pi_2^*, \Pi_2)$ topology. We call this topology the weak-* topology. We have thus proven the following lemma

Lemma 3.1 *The set* $\{\Phi \in W^{2,\infty}(U); ||\Phi - Id||_{W^{2,\infty}} \leq k\}$ *is closed and compact in the weak-* topology of* Π_2^*.

In order to be able to use our Sobolev inequalities derived in chapter 2 we need to know that $||\Phi^k - v||_{W^{1,\infty}(U)}$ goes to zero when Φ^k goes to v weakly-*. It is well known that $W^{2,\infty}$ is compactly imbedded in $W^{1,\infty}$. This follows from the fact that $W^{s,\infty}$ can be identified with $C^{s,1}$, and that the Arzela-Ascoli Theorem (see Appendix) says that $C^{s,1}$ is compactly imbedded in $C^{s-1,1}$. This means that we can refine the sequence above to get that it converges to v' in $W^{1,\infty}$ strongly. Since we only refine the sequence it will still converge to Φ_∞ in the $\sigma(\Pi_2^*, \Pi_2)$ topology. We now look at the jet-space

$$\Pi_1^* = \underbrace{L^\infty(U) \times L^\infty(U) \times ...L^\infty(U)}_{n+1}$$

with norm

$$||f||_{\Pi_1^*} = ||(f_1, ..., f_n)||_{\Pi_1^*} = \max(||f_1||_{L^\infty(U)}, ..., ||f_n||_{L^\infty(U)})$$

The mapping

$$\Phi \mapsto (\Phi, \partial_1\Phi, ..., \partial_n\Phi)$$

is an isometry from $W^{1,\infty}$ to Π^*. If Φ^k converges to v' in $W^{1,\infty}$ strongly then it will also converges to v' in the $\sigma(\Pi_1^*, \Pi_1)$ topology. Now since convergence in the $\sigma(\Pi_2^*, \Pi_2)$ topology implies convergence in the $\sigma(\Pi_1^*, \Pi_1)$ topology, we have that Φ^k converges to both v and v' in the $\sigma(\Pi_1^*, \Pi_1)$ topology and therefore we must have that $v = v'$. This means that the following lemma is true

Lemma 3.2 *The set* $\{\Phi \in W^{2,\infty}(U); ||\Phi - Id||_{W^{2,\infty}} \leq k\}$ *is closed and compact in the strong topology of* $W^{1,\infty}(U)$.

3.1.1 Convergence in $\mathcal{A}$

We will now show that if Φ^k is in $\mathcal{A}$, we can find a subsequence Φ_j^k which converges in $\mathcal{A}$. Recall that the definition of $\mathcal{A}$ is

$$\mathcal{A} = \{\Phi \in W^{2,\infty}(U); ||\Phi - Id||_{W^{2,\infty}(U)} \leq k, ||\Phi' - I||_2 \leq 1/2,$$

$$\Phi_1(x) = x_1, \Phi(x) = x \ if \ x \in F, \int_{\Phi(B)} dx = \int_B dx\}$$

Lemma 3.1 tells us that if Φ^k fulfills the condition $||\Phi^k - Id||_{W^{2,\infty}(U)} \leq k$ then we can find a sequence Φ^{k_j} that converges weakly-* to a Φ^∞ that fulfills $||\Phi^k - Id||_{W^{2,\infty}(U)} \leq k$. Lemma 3.2 tells us that this sequence can be chosen such that $||\Phi^{k_j} - \Phi^\infty||_{W^{1,\infty}(U)} \to 0$ as $k_j \to 0$.

The mapping $M_1 : W^{1,\infty}(U) \ni \Phi^k \mapsto ||(\Phi^k)' - I||_2 \in \mathbb{R}_+$ is continuous and since $||\Phi^{k_j} - \Phi^\infty||_{W^{1,\infty}(U)} \to 0$ as $k_j \to 0$ we get that

$$||(\Phi^\infty)' - I||_2 \leq 1/2$$

Recall that the operator norm $||\cdot||_2$ is defined by

$$||\Phi'||_2 = \sup_{|v|_2=1} |\Phi' v|_2$$

where $|\cdot|_2$ is the standard euclidian absolute value.

For a fixed $x \in U$ the mapping $M_2 : C^{0,1} \in \Phi^k \to \Phi_1^k(x) = \Phi_1^k(x) - x_1 \in \mathbb{R}$ is continuous. Since $M_2(\Phi^k) = 0 \ \forall k$ we have that $M_2(\Phi^\infty) = 0$, that is $\Phi_1^\infty(x) = x_1$ if $x \in U$. Exactly the same argument gives us that $\Phi^\infty(x) = x$ if $\Phi \in F$.

Finally for the last condition. We have that

$$\int_{\Phi(B)} dx = \int_B |\det \Phi'| dx$$

The mapping

$$M_3 : W^{1,\infty} \ni \Phi \mapsto \int_B |\det \Phi'| dx - \int_B dx \in \mathbb{R}$$

is continuous and since $F(\Phi^k) = 0 \ \forall k$ it follows that

$$\int_{\Phi^\infty(U)} dx = \int_B dx$$

This gives us our compactness lemma.

Lemma 3.3 *The set $\mathcal{A}$ is compact in the $W^{1,\infty}$-norm topology.*

3.1.2 The sublevel-sets $\mathcal{A}_\alpha$

In order to be able to show existence of solutions to our problem we need to show that the sublevel-sets $\mathcal{A}_\alpha = \{\Phi \in \mathcal{A}; E(\Phi) \leq \alpha\}$ are closed in $W^{1,\infty}$, and that at least one of them is compact and not empty. Since we now that a closed subset of a compact set is closed the compactness will follow from the fact that they are closed.

That $\mathcal{A}_\alpha = \{\Phi \in \mathcal{A}; E(\Phi) \leq \alpha\}$ is closed means that the pre image of $[0, \alpha]$ is closed. But this is true since by theorem 2.3, corollary 2.4 and corollary 2.5 we have that I is continuous on $\mathcal{A}$, and therefore by definition the pre image is closed. This gives us our compactness lemma

Lemma 3.4 *The sublevel-sets $\mathcal{A}_\alpha = \{\Phi \in \mathcal{A}; E(\Phi) \leq \alpha\}$ are closed and compact in $W^{1,\infty}(U)$.*

3.1.3 Existence of solutions

Everything is now in place for our existence theorem. Due to Theorem A.1 we have that Problem 1,2 and 3 has a solution. $\Leftrightarrow$ There exists a topology on $\mathcal{A}$ such that

1. For all $\alpha \in \mathbb{R}$ the sublevel-sets $\mathcal{A}_\alpha$ are closed.
2. There exists an $\alpha_0 \in \mathbb{R}$ such that $\mathcal{A}_{\alpha_0}$ is non-empty an compact.

The topology on $\mathcal{A}$ is the strong topology of $W^{1,\infty}$, and as we have seen the sublevel-sets $\mathcal{A}_\alpha$ are closed and compact with this topology. Obviously not all of the sublevel-sets are empty since if we choose $\Phi = Id$ then $E(\Phi) = ||\nabla u||^2_{L^2(D)}$ which is finite.

Theorem 3.1 *Problem 1,2 and 3 has a solution.*

Remark. Note that for problem 3 we have to demand that the Reynolds number is small enough such that Theorem 2.7 is fulfilled.

Chapter 4

Numerics

In the previous chapters we have been concerned with showing that our problem has a solution. In this chapter will now briefly study how we can find it by using femlab and matlab. To be able to prove that a local minimizer is in fact a global minimizer, one needs to have a function which has some special property like e.g. convexity. Unfortunately this does not seem to be the case with our problem. Therefore we will only be able to look for local minimas by steepest decent methods. However by selecting sufficiently many starting points we hope to find all local minimas, and then determine which one is the global minima.

4.1 Solution of the problem with matlab/femlab

Now that we know from chapter 3 that there is a solution to the problem the obvious way to proceed would be to say that, if we have two domains which are close to each other in some sense, then there is a mapping between the two which is close to the identity. We could then construct domains which depend on some parameters and solve with femlab. Since it is much easier to construct these domains than to construct mappings which fulfill all conditions in $\mathcal{A}$, this seems like a good idea. However this approach has a downside. Since we are going to use steepest decent methods to find the minimum we need to be able to calculate derivatives, in our case, numerically. This means that we will make very small changes to the domain and evaluating what happens. However since we use femlab a small change to the domain might lead to a completely different mesh. This seems to make the steepest decent method produce very strange results. One would think that a very fine mesh would limit this problem however due to limitations in computer power we can not make the mesh as small as we need to. Therefore we again turn to our mappings. If we have mapping from one domain to another, close to the identity, we can apply it to the mesh an get a new mesh which is almost the same as the old one. This gives us a much nicer goal function to evaluate.

4.1.1 Construction of mappings

We will now discuss how to construct a family of very simple mappings in $\mathbb{R}^2$. The problem with these mappings is the invertibility demand. We cannot simply construct a base of mappings and then add them together because the addition may destroy the invertibility. A number of methods have been tested like e.g. using B-splines and using their control points as parameters. However this gives very complicated expressions for the volume change. The method below is the one that gives the simplest calculations and the one we have used.

We want the mappings to have the form $\Phi(x, y) = (x, \Phi_2(x, y) + y)$, because if Φ_2 is small this will be close to $Id(x, y) = (x, y)$. We start by looking at a line L parallel to the y-axis. We want to move a point d on the length h in the direction of L. The idea is simply to take a function f twice differentiable, with shape like in figure 4.1.

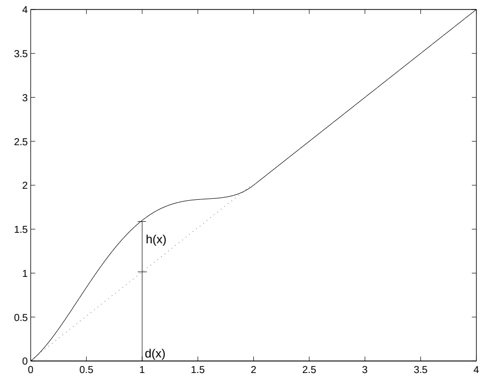

Figure 4.1: Mapping which moves the point d(x) the length h(x).

Since we demand that the mappings should be invertible we get that $\frac{\partial \Phi_2}{\partial y}$ should be less than 1. This gives us a bound on h. If we let d and h depend on x, then $d(x)$ will be the curve that we want to move and $h(x)$ will be the length which it will move. Φ will now have the form

$$\Phi(x, y) = (x, h(x)f(d(x), y) + y)$$

We note that the change of the area between the x axis and the curve will be

$$\int h(x)\,dx.$$

We can now have some parameters k to control h. In our case we used three or five parameters which controls the value of h in three or five intervals. In our case we made the construction so that the constant area condition gives a linear condition on k. The invertibility condition will also give a bound on the size of k. If we apply these mappings to the mesh we get quite good results see figure 4.2.

Using femlab we can now evaluate the goal function, and using matlabs own functions for optimization we can minimize it with respect to the parameters k.

4.2 Results

We made the calculation for four different cases. One for the Stokes version of the problem and three for Navier-Stokes with different Reynolds number. The

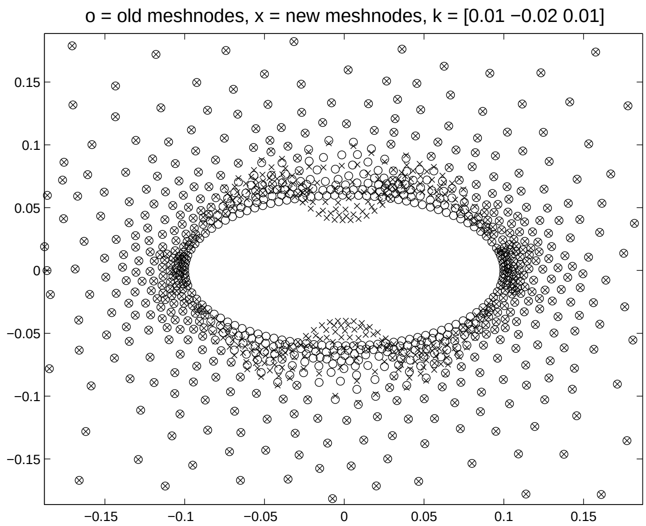

Figure 4.2: The nodes of the mesh before and after applying Φ.

trouble with Navier-Stokes equations is that when the Reynolds number is high the flow becomes turbulent, which makes it very hard solve, numerically. Air flow is very turbulent, therefore we had to limit ourselves to flows with much lower Reynolds number.

In the case with three parameters we can actually plot the goal function since we have a linear condition on k which gives the last parameter, see figures 4.3-4.6. Each of these goal functions contain 109 data points, and it took about a day for the computer to calculate them, plus a number of failed attempts.

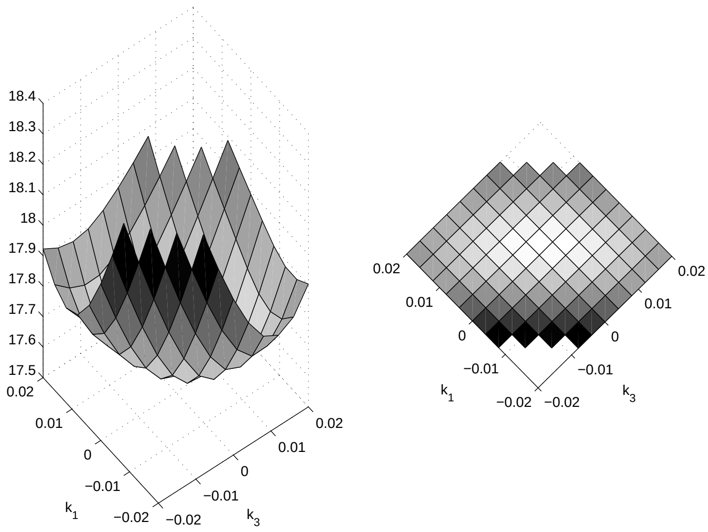

Figure 4.3: Goal function for the problem stated with the Stokes equations.

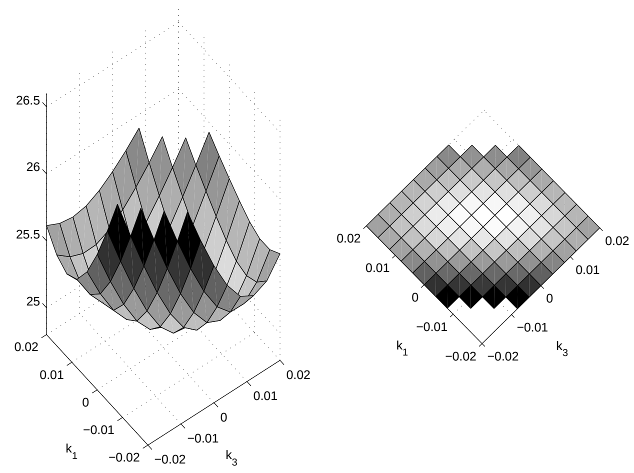

Figure 4.4: Goal function for the problem stated with the Naiver stokes equations with Reynolds number 1.

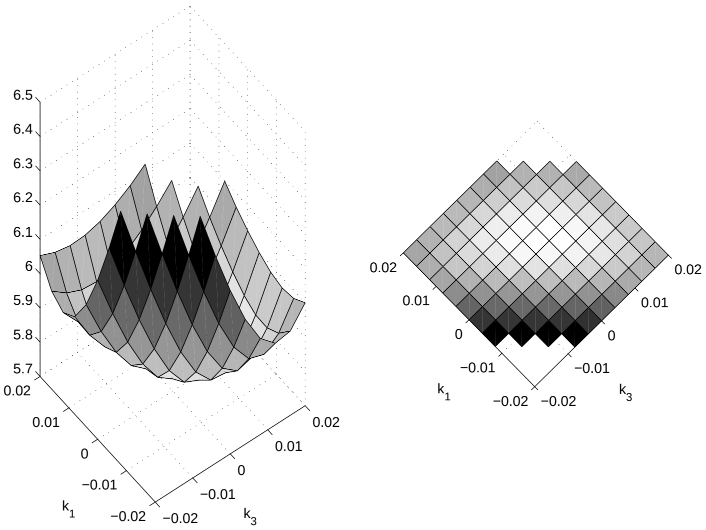

Figure 4.5: Goal function for the problem stated with the Navier-Stokes equations with Reynolds number 10.

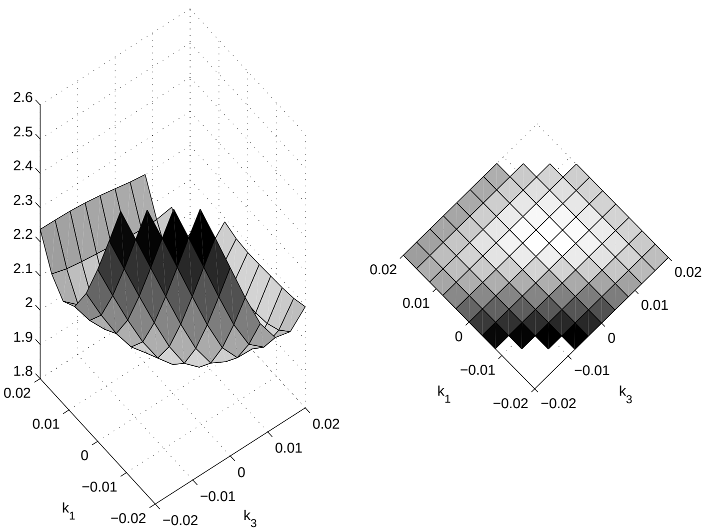

Figure 4.6: Goal function for the problem stated with the Navier-Stokes equations with Reynolds number 100.

The minimum seems to be close to zero in all four cases. The table below shows the result of the steepest decent search.

		Starting point:	Stokes	Re = 1	Re = 10	Re = 100
	k1	0	-0.0010	-0.0008	-0.0007	-0.0005
	k2	0	0.0006	0.0003	-0.0011	-0.0028
	k3	0	0.0004	0.0005	0.0018	0.0033
	k1	0.02	-0.0010	-0.0008	-0.0007	-0.0005
	k2	0	0.0006	0.0003	-0.0011	-0.0028
	k3	-0.02	0.0004	0.0005	0.0018	0.0033
2	k1	-0.02	-0.0010	-0.0008	-0.0007	-0.0005
	k2	0	0.0006	0.0003	-0.0011	-0.0028
	k3	0.02	0.0004	0.0005	0.0018	0.0033
	k1	-0.02	-0.0010	-0.0008	-0.0007	-0.0005
	k2	0.03	0.0006	0.0003	-0.0011	-0.0029
	k3	-0.01	0.0004	0.0005	0.0018	0.0033
	k1	0.01	-0.0010	-0.0008	-0.0007	-0.0005
	k2	-0.02	0.0006	0.0003	-0.0011	-0.0029
	k3	0.01	0.0004	0.0005	0.0018	0.0033

Figures 4.7 - 4.10 shows the flow, with Reynolds number 100, around four different shaped obstacles. The shape in figure 4.8 is the one closest to the optimal shape. If one looks carefully one can see that the black area behind the obstacle in 4.8 is slightly smaller than in the other pictures. The black area is in fact particles which have lower velocity. This means that this obstacle affects the flow behind it less than the other ones. We also see that the area in front of the obstacle is quite similar in all pictures. All this would indicate that the shape in figure 4.8 is the best of these four.

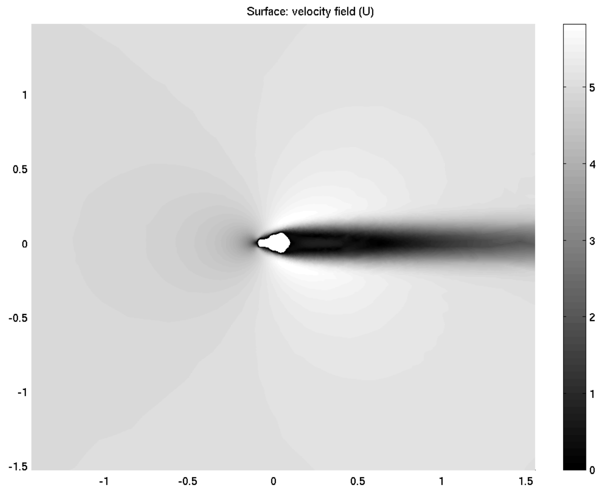

Figure 4.7: Flow around a shape with k = (-0.02, 0, 0.02) (Re = 100).

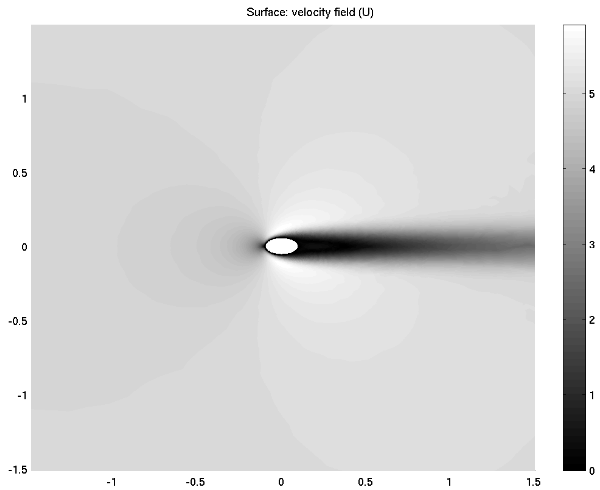

Figure 4.8: Flow around a shape with k = (0, 0, 0) (Re = 100).

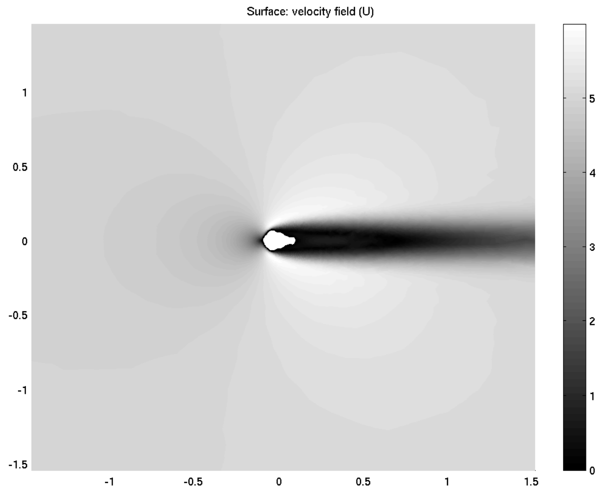

Figure 4.9: Flow around a shape with k = (-0.02, 0, 0.02) (Re = 100).

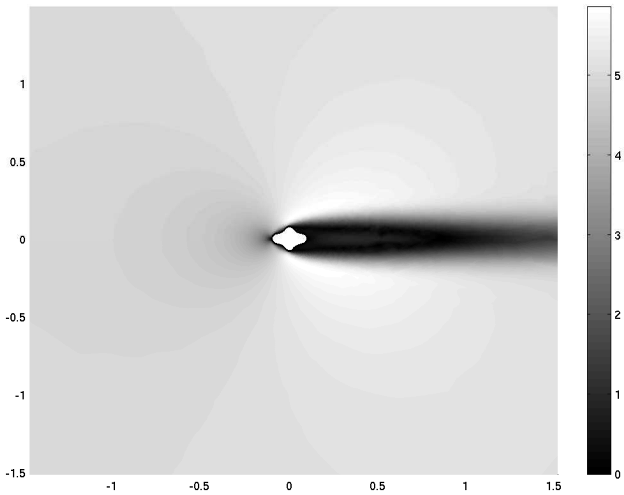

Figure 4.10: Flow around a shape with k = (-0.02, 0, 0.02) (Re = 100).

In the case where five parameters were used we got the following table (which took about five days to compute).

	Starting point:	Stokes	Re = 1	Re = 10	Re = 100
k1	0	-0.0019	-0.0015	-0.0003	0.0009
k2	0	-0.0001	-0.0003	-0.0017	-0.0013
k3	0	0.0006	0.0003	-0.0016	-0.0040
k4	0	0.0006	0.0004	0.0000	-0.0018
k5	0	0.0009	0.0011	0.0036	0.0061
k1	-0.01	-0.0019	-0.0015	-0.0003	0.0008
k2	0	-0.0001	-0.0003	-0.0017	-0.0031
k3	0	0.0006	0.0003	-0.0016	-0.0037
k4	0	0.0005	0.0004	0.0000	-0.0011
k5	0.01	0.0009	0.0011	0.0036	0.0070
k1	-0.01	-0.0019	-0.0015	-0.0019	0.0009
k2	0.01	-0.0001	-0.0003	-0.0001	-0.0031
k3	0	0.0006	0.0003	-0.0006	-0.0037
k4	0.01	0.0005	0.0004	0.0006	-0.0011
k5	-0.01	0.0009	0.0011	0.0009	0.0070
k1	-0.01	0.0045	-0.0011	-0.0009	-0.0001
k2	0	0.0069	0.0006	-0.0012	-0.0025
k3	0.02	0.0041	0.0009	-0.0013	-0.0038
k4	0	-0.0055	0.0003	0.0001	-0.0009
k5	-0.01	-0.01	-0.0007	0.0034	0.0072
k1	-0.01	-0.0003	0.01	0	-0.0001
k2	-0.01	0.0014	-0.0200	0.0050	-0.0024
k3	-0.01	0.0014	0.0053	-0.0150	-0.0038
k4	0.02	0.0001	0.0036	0.0000	-0.0009
k5	0.01	-0.0026	0.0011	0.0100	0.0072
k1	0.01	-0.0100	-0.0011	-0.0009	-0.0001
k2	-0.0067	0.0064	0.0006	-0.0012	-0.0024
k3	-0.0067	0.0200	0.0009	-0.0013	-0.0038
k4	-0.0067	-0.0064	0.0003	0.0001	-0.0009
k5	0.01	-0.0100	-0.0007	0.0034	0.0072
k1	0.01	0.0100	-0.0048	-0.009	0.0007
k2	0.02	0	-0.0061	-0.0012	-0.0024
k3	-0.01	0.0050	0.0043	-0.0013	-0.0045
k4	-0.01	-0.0050	0.0079	0.001	-0.0011
k5	-0.01	-0.0100	-0.0013	0.0034	0.0073
k1	0.01	0.0100	-0.0011	-0.0009	-0.0001
k2	0.02	0.0000	0.0006	-0.0012	-0.0025
k3	0	-0.0000	0.0009	-0.0013	-0.0038
k4	-0.02	0	0.0003	0.0001	-0.0009
k5	-0.01	-0.0100	-0.0007	0.0034	0.0072

The next table shows the corresponding values of the drag.

Stokes	Re = 1	Re = 10	Re = 100
17.5322	24.9341	5.7306	1.8571
17.5323	24.9341	5.7306	1.8564
17.5323	24.9341	5.7364	1.8565
17.5983	24.9306	5.7300	1.8556
17.5260	25.3139	5.8553	1.8556
17.9421	24.9306	5.7300	1.8556
17.6911	25.0688	5.7300	1.8561
17.6421	24.9306	5.7300	1.8556

It is difficult to say anything from these values. Since the most of the starting points give different minima we would need lots of more starting points. However we see that in the last case with Reynolds number 100, all the minimas have a similar shape. It is therefore tempting to assume that the minimum is close to these. The best shapes we have found are shown in figures 4.11-4.14.

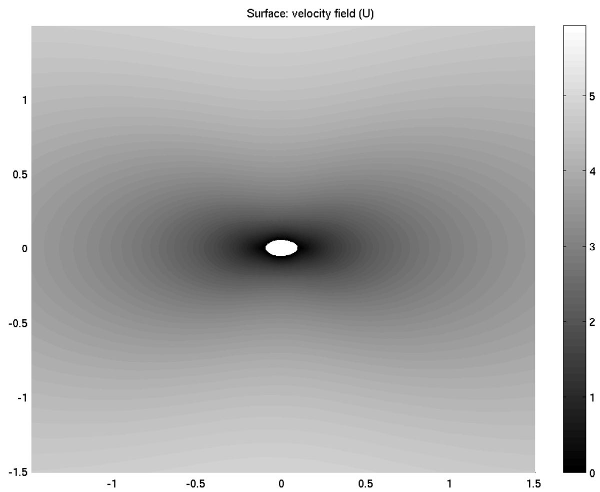

Figure 4.11: k = (-0.0003, 0.0014, 0.0014, 0.0001, -0.0026). (Stokes)

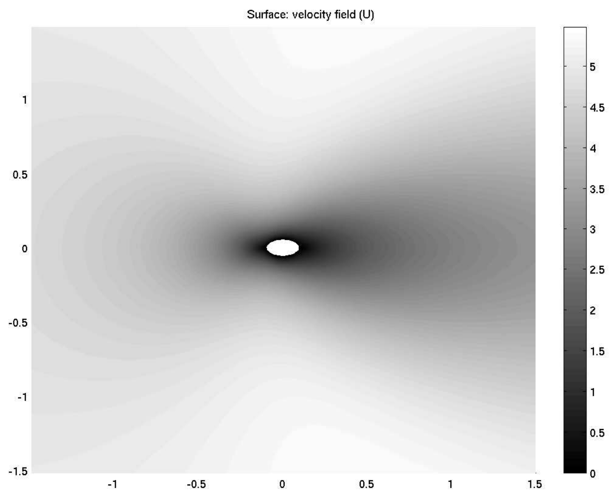

Figure 4.12: k = (-0.0011, 0.0006, 0.0009, 0.0003, -0.0007). (Re = 1)

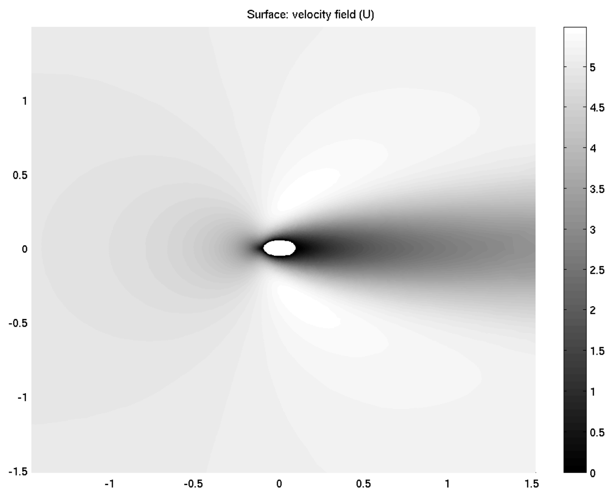

Figure 4.13: k = (-0.0009, -0.0012, -0.0013, 0.0001, 0.0034). (Re = 10)

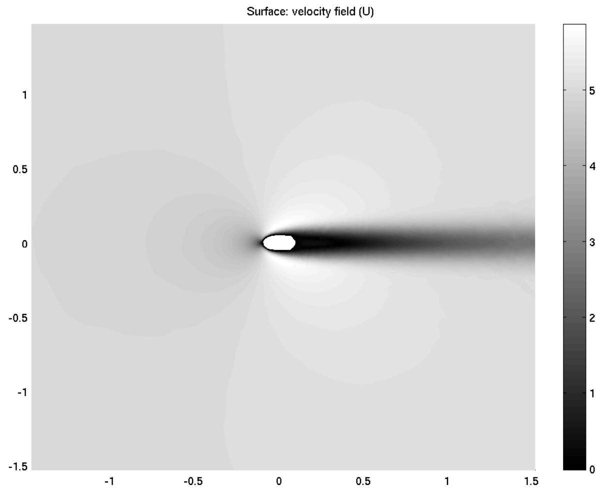

Figure 4.14: k = (-0.0001, -0.0025, -0.0038, -0.0009, 0.0072). (Re = 100)

Figure 4.15 shows the flow around an obstacle shaped like a drop of water. One would expect that this shape would be optimal. However the our obstacle is not affected in the same way by gravity as a drop of water. Surprisingly we get a rather high drag for the drop shape. For the drop we get 2.0241 while for our best shape we get 1.8556. Indeed if we compare figures 4.14 and 4.15 we can see that the drop shaped obstacle affects the flow slightly more than our best shape. This strange result might be because of the lower Reynolds number.

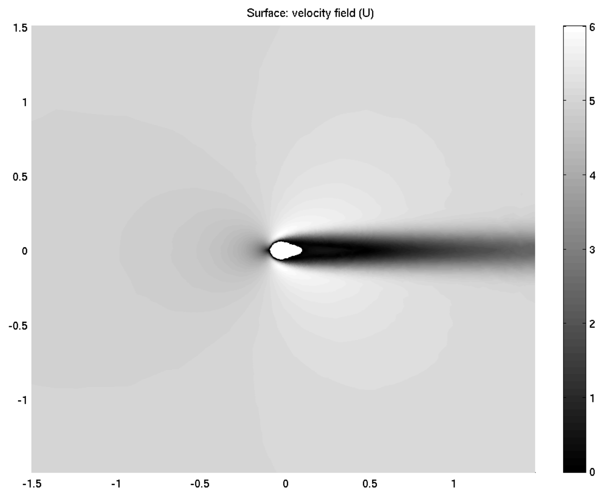

Figure 4.15: k = (0.01, 0.01, 0, -0.01, -0.01). (Re = 100)

Appendix A

General Theoretical Background.

In this appendix we will provide a very short theoretical background for the concepts used in this thesis. For a more complete background of the basic material [8] and [1] is worth consulting, and for the material concerning Stokes and Navier-Stokes equations [7] is recommended. For a derivation of Navier-Stokes equations see [9].

A.1 General variational problems

The theoretical part of this thesis is devoted to proving that problems 1,2 and 3 (defined in sections 2.1, 2.2 and 2.3) has a solution. This can be done thanks to the following theorem which is taken from [5].

Problem A.1 *Is there a* $\Phi_0 \in \mathcal{S}$ *such that*

$$E(\Phi_0) \leq E(\Phi), \quad \forall \Phi \in \mathcal{S}$$

Theorem A.1 *Problem A.1 has a solution if and only if there exists a topology on* $\mathcal{S}$ *such that*

1. *For all* $\alpha \in \mathbb{R}$ *the sublevel sets*

   $$\mathcal{S}_\alpha = \{\Phi \in \mathcal{S}; E(\Phi) \leq \alpha\}$$

   are closed.

2. *There exists an* $\alpha_0 \in \mathbb{R}$ *such that* $\mathcal{S}_{\alpha_0}$ *is closed an compact.*

Proof We will only prove "if" an not "if and only if" since this is all that is used in this thesis. Take a non-empty and compact $\mathcal{S}_{\alpha_0}$. Then for all $\alpha \leq \alpha_0$ the sets $\mathcal{S}_\alpha$ are the intersection of a closed and a compact set and are thus compact. Now let $m = \inf_{\Phi \in S} E(\Phi)$ and pick a decreasing sequence $\{\alpha_i\}_{i=1}^{\infty}$ tending to m with $\alpha_i \leq \alpha_0$ such that $\mathcal{S}_{\alpha_i}$ are all non empty. Any point in this intersection of $\{E_{\alpha_i}\}_{i=1}^{\infty}$ is a solution to Problem A.1.

A.2 Spaces

In this section we introduce some of the different spaces which are used in this thesis. The standard way to define distance between functions is to use a normed vector space. A normed vector space is called a Banach space if it is complete in the metric coming from the norm, i.e. if every sequence $\{x_n\}$, such that $\lim_{m,n\to\infty} ||x_n - x_m|| = 0$, has a limit. If $D \subset \mathbb{R}^n$ we make the following definition

Definition A.1 *The space $L^p(D)$ $1 \leq p < \infty$ is the space of real valued functions such that $||u||_{L^p(D)} \leq \infty$, where*

$$||u||_{L^p(D)} = \left(\int_D |u|^p \, dx \right)^{1/p}$$

The space $L^\infty(D)$ is the space of real valued functions such that $||u||_{L^\infty(D)} = \lim_{p\to\infty} ||u||_{L^p(D)} < \infty$.

The $L^p(D)$ spaces are a Banach spaces with the corresponding norm $|| \cdot ||_{L^p(D)}$. The norm $|| \cdot ||_{L^\infty(D)}$ can also be written as

$$||u||_{L^\infty(D)} = \text{ess.} \sup_{x\in D} |u(x)|$$

If the function u is continuous then

$$||u||_{L^\infty(D)} = \sup_{x\in D} |u(x)|.$$

Remark. Note that when dealing with L^p spaces we have to redefine what it means that two functions are equal. We say that two functions are equivalent $(f \equiv g)$ if they only differ on a set of measure zero. A more correct statement is then that the space $L^p/\equiv$ is a Banach space. It is easy to see that without this equivalence relation the L^p spaces will not be Banach. In fact the L^p norm will not be a norm since any function which is zero everywhere except on a set of measure zero will have L^p-norm zero.

The analysis of partial differential equations involves not only the function values but also the values of the derivatives. So if we want to ensure convergence for the derivatives we need to include them in the norms. This leads to the Sobolev spaces.

Definition A.2 *The space $W^{k,p}(D)$ is the set of all functions $u \in L^p(D)$ such that the derivatives of order k is in $L^p(D)$. In $W^{k,p}(D)$ we define the norm by*

$$||u||_{W^{k,p}(D)} = \left(\sum_{|\alpha|\leq k} ||\partial^\alpha u||^p_{L^p(D)} \right), \ p \leq \infty$$

$$||u||_{W^{k,\infty}(D)} = \max_{|\alpha|\leq k} ||D^\alpha u||_{L^\infty(D)}$$

The space $W^{k,p}(D)$ is a Banach space, and if $p \leq \infty$ it is also separable. The space $W^{k,2}(D)$ is usually denoted H^k. If k is not an integer then $W^{k,p}(D)$ can be defined via the Fourier transform.

Definition A.3 *The space $W_0^{k,p}(D)$ is the closure of the test functions $\mathcal{D}(D)$ in $W^{k,p}(D)$-norm. Similary $H_0^k(D)$ is the closure of the testfunctions in $H^k(D)$-norm.*

$W_0^{k,p}(D)$ is obviously a Banach space. Sometimes it is convenient to use the norm defined below on H_0^1.

Definition A.4 $||u||_{H_0^1(D)} = ||\nabla u||_{L^2(D)}$

A.3 Poincaré's inequality

A very useful lemma is Poincaré's inequality. It tells us that if the domain is bounded then the functions in $H_0^1(D)$ are bounded by their derivatives. This is essential for proving existence of solutions for Laplace's equation. It is used to show that $\int_D \nabla u \cdot \nabla v \, dx$ is an inner product on $H_0^1(D)$.

Lemma A.1 *Let D be contained in the strip $|x_1| \leq d < \infty$. Then there is a constant c, depending only on k and d, such that*

$$||u||_{H^k(D)}^2 \leq c \sum_{|\alpha|=k} ||D^\alpha u||_{L^2(D)}^2$$

for every $u \in H_0^1(D)$.

Proof. We give the proof for k = 1, the general case then follows by induction. We take $u \in \mathcal{D}(D)$. Integration by parts yields

$$||u||_{L^2(D)}^2 = \int_D 1|u(x)|^2 \, dx = -\int_D x_1 \frac{\partial}{\partial x_1}|u|^2 \, dx \leq 2d||u||_{L^2(D)}||\frac{\partial u}{\partial x_1}||_{L^2(D)}$$

Since $\mathcal{D}(D)$ is dense in $H_0^1(D)$ this holds for all $u \in H_0^1(D)$.

A.4 The trace theorem

Due to [1] we have a theorem that can be used to get information on how solutions of partial differential equations depend on their boundary values. We will give the theorem here without proof.

Theorem A.2 *Let $s \geq 1/2$. There exists a continuous linear map $T : H^s(\mathbb{R}^m) \to H^{s-1/2}(\mathbb{R}^{m-1})$, called the trace operator, with the property that for any $\phi \in \mathcal{D}(\mathbb{R}^n)$ we have*

$$T\phi(x_1, \ldots, x_{m-1}) = \phi(x_1, \ldots, x_{m-1}, 0).$$

The theorem is proven by putting $g(x_1, \ldots, x_{m-1}) = \phi(x_1, \ldots, x_{m-1}, 0)$ and noting that

$$\mathcal{F}g(\xi_1, ..., \xi_{m-1}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}\phi(\xi_1, ..., \xi_m) d\xi_m.$$

$||g||_{H^{s-1/2}(D)}$ can then be estimated with the Fourier version of the norm. We also have a theorem that gives us an inverse operator, the extension operator, to the trace operator.

Theorem A.3 *Let $s \leq 1/2$. Then there exists a bounded linear mapping $E : H^{s-1/2}(\mathbb{R}^{m-1}) \to H^s(\mathbb{R}^m)$, such that $T \circ E = Id$.*

The proof is obtained by explicit construction of E.

$$E\phi = \frac{1}{c} \int_{\mathbb{R}^m} e^{i\xi \cdot x} \frac{(1 + |\xi'|^2)^{s-1/2}}{(1 + |\xi|^2)^s} \mathcal{F}\phi d\xi$$

where $\xi' = (\xi_1, ..., \xi_{m-1})$.

By using a partition of unity argument these results can be extended to bounded domains with C^k boundary.

A.5 The Arzela-Ascoli Theorem

The following theorem is due to [1]. In this thesis we use it to prove that $W^{2,\infty}$ is compactly imbedded in $W^{1,\infty}$.

Theorem A.4 (Arzela-Ascoli) *Let $\{f_m\}$ be a sequence of real valued functions defined in a compact subset S of $\mathbb{R}^n$. Assume that there is a constant M such that $|f_m(x)| \leq M$ for every $m \in \mathbb{N}$ and every $x \in S$. Moreover assume that the sequence f_m is equicontinuous at every point of S. Then there exists a subsequence which converges uniformly on S.*

Proof. Let $\{x_i\}$, $i \in \mathbb{N}$ be a sequence of points that is dense in S The sequence $\{f_m(x_1)\}$ is bounded, thus we can choose a subsequence m_{1j}, such that $\{f_{m_{1j}}(x_1)\}$ converges as $j \to \infty$. Similarly, we can choose a subsequence $m_2 j$ of m_{1j} such that $\{f_{m_{2j}}(x_2)\}$ converges as well. We do this for all x_i. Now we consider the diagonal sequence $\{f_{m_{jj}}\}$. Except for the first $i-1$ terms, m_{jj} is a subsequence of m_{ij}, and therefore $\{f_{m_{jj}}(x_i)\}$ converges for every $i \in \mathbb{N}$. Now put $g_j = f_{m_{jj}}$. Since $\{g_m\}$ is a subsequence of f_m it is equicontinuous on S, therefore for every $\epsilon > 0$ there is a $\delta > 0$ such that $|g_m(y) - g_m(x)| < \epsilon/3$ whenever $|y - x| < \delta$. Since S is compact there is a $K \in \mathbb{N}$ such that for every $x \in S$ there exists $i \in \{1, ..., K\}$ with $|x_i - x| < \delta$. Now we choose N large enough such that $|g_m(x_i) - g_k(x_i)| < \epsilon/3$ for $m, k > N$ and every $i \in \{1, ..., K\}$. For m,k $>$ N and arbitrary $x \in S$ we now have

$$|g_m(x) - g_k(x)| \leq |g_m(x) - g_m(x_i)| + |g_m(x_i) - g_k(x_i)| + |g_k(x_i) - g_k(x)| \leq \epsilon$$

for some $i \in \{1, ..., K\}$. Therefore the sequence is uniformly Cauchy.

A.6 The Banach-Alaogu Theorem

The following theorem is also form [1]. It is used to prove that a bounded set in $W^{2,\infty}$ is weakly* compact.

Theorem A.5 (Banach-Alaogu) *Let X be a separable Banach space and let f_n be a bounded sequence in X^*. Then f_n has a weakly convergent subsequence.*

Proof. Let $\{x_k\}_{k\in(N)}$ be dense in X. We start by using a standard diagonal argument to get a sequence that converges for all x_k since f_n is bounded. Let M be an upper bound for $||f_n||$. To see that the sequence converges for every x we do the following. For a given $\epsilon > 0$ and arbitrary $x \in X$, we can choose k such that $||x - x_k|| \leq \epsilon/(3M)$. We then have

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(x_k)| + |f_n(x_k) - f_m(x_k)| + |f_m(x_k) - f_m(x_k)| \leq \epsilon$$

if m and n are large enough.

A.7 The governing equations

In this section we will look at the weak formulation of our governing equations. The reason for using weak formulations is that it is easier to prove that there is a solution to the weak problem and that under certain conditions this solution also solves the strong problem, then proving directly that there is a solution to the strong problem.

A.7.1 Laplace's Equation

The homogenous case

We begin by looking at the homogenous Laplace equation. The non homogenous case can be transformed into the homogenous by using the trace theorem. The equations are

$$\begin{cases} -\Delta u &= f \quad D \\ u &= 0 \quad \partial D \end{cases}$$

In the distribution sense this is equivalent to finding a u such that

$$-\int_D \Delta u\, \phi\, dx = \int_D f\phi\, dx \qquad \forall \phi \in \mathcal{D}(D)$$

where u is zero on ∂D in some sense. Since u is zero on ∂D we have due to Gauss Theorem that

$$\int_D \Delta u\, \phi\, dx = \int_D \nabla \cdot (\nabla u \phi)\, dx - \int_D \nabla u \cdot \nabla \phi\, dx = -\int_D \nabla u \cdot \nabla \phi\, dx$$

The weak formulation is thus

$$\int_D \nabla u \cdot \nabla \phi\, dx = \int_D f\phi\, dx \tag{A.1}$$

where $\phi \in H_0^1(D)$, since $\mathcal{D}(D)$ is dense in $H_0^1(D)$. Put $B(u,\phi) = \int_D \nabla u \cdot \nabla \phi\, dx$ and $L(\phi) = \int_D f\phi\, dx$. In order to prove existence and uniqueness of solutions one can now prove that $B(u,\phi)$ is an inner product on $H_0^1(D)$. This follows from the fact that $B(u,\phi)$ is continuous and equivalent to the standard inner product on $H^1(D)$

$$(u,v)_{H^1} = \int_D (uv + \nabla u \cdot \nabla v)\, dx$$

Lemma A.2 *$B(u,\phi)$ is continuous on $H_0^1(D) \times H_0^1(D)$.*

Proof. If $u \in H^1, \phi \in \mathcal{D}(D)$ we have that

$$||B(u,\phi)|| \leq \int_D |\nabla u \cdot \nabla \phi|\, dx \leq \int_D |\nabla u||\nabla \phi|\, dx \leq$$

$$(\int_D |\nabla u|^2\, dx)^{1/2}(\int_D |\nabla \phi|^2\, dx)^{1/2} \leq ||\nabla u||_{L^2(D)}||\nabla \phi||_{L^2(D)}$$

Since $\mathcal{D}(D)$ is dense in $H_0^1(D)$, B is continuous in $H_0^1(D) \times H_0^1(D)$.

Lemma A.3 *$B(u,\phi)$ is an inner product on $H_0^1(D)$ equivalent with the standard inner product.*

Proof. If $u \in \mathcal{D}(D)$ we have due to Poincaré's lemma

$$B(u,u) = \int_D |\nabla u|^2\, dx = \frac{1}{2}\int_D |\nabla u|^2\, dx + \frac{1}{2}\int_D |\nabla u|^2\, dx$$

$$\geq \frac{1}{2}\int_D |\nabla u|^2\, dx + \frac{1}{2k}\int_D |u|^2\, dx \geq c||u||_{H^1}^2$$

Schwarz inequality now gives

$$c||u||_{H^1}^2 \leq B(u,u) \leq C||u||_{H^1}^2$$

The fact that $\mathcal{D}(D)$ is dense in $H_0^1(D)$ proves the lemma.

We now use Reisz representation theorem which says that, if H is a Hilbert space and $L : H \mapsto R$ is a linear continuous functional on H then there is a unique $u \in H$ such that $(u,\phi) = L(\phi)\ \forall \phi \in H$. The fact that $L(\phi)$ is continuous follows from the fact that

$$\left|\int_D f\phi\, dx\right| \leq ||f||_{L^2}||\phi||_{L^2} \tag{A.2}$$

This proves that there is a solution to the weak problem. (A.1) and (A.2) also gives us a regularity result

$$||u||_{H^1(D)} \leq c||f||_{L^2(D)}$$

In fact a better regularity result can be obtained. A classical result is, if ∂D is in C^s we have

$$||u||_{H^s(D)} \leq c||f||_{H^{s-2}(D)} \tag{A.3}$$

where s is an integer (see [12]).

The non homogenous case

We assume that ∂D is in C^k ($k \in \mathbb{N}$) and ∂D is bounded. Due to the Trace Theorem (see [1] or Appendix) we know that if $s \leq k$ there exists a bounded linear trace operator $T : H^s(D) \to H^{s-1/2}(\partial D)$, and a bounded linear extension operator $E : H^{s-1/2}(\partial D) \to H^s(D)$ such that $T \circ E$ is the identity. Using the extension operator we can trade the homogeneous force terms in 2.1 for boundary data. If we put $F = Ef$ and $u = w + F$ we get

$$\left\{ \begin{array}{rcll} \Delta(w+F) & = \Delta u = & 0 & in\ D \\ w+f & = \ u \ = & 0 & on\ \partial D \end{array} \right. \Rightarrow \left\{ \begin{array}{rcll} \Delta w & = & -\Delta F & in\ D \\ w & = & 0 & on\ \partial D \end{array} \right.$$

In the weak formulation we get

$$\int_D \nabla w \cdot \nabla \phi \, dx = - \int_D \nabla F \cdot \nabla \phi \, dx \quad \forall \phi \in \mathcal{D}(D)$$

Since $\mathcal{D}(D)$ is dense in $H_0^1(D)$ this extends to all $\phi \in H_0^1(D)$. Setting $\phi = w$ and applying Cauchy-Schwarz inequality yields

$$\int_D |\nabla w|^2 \, dx = \int_D \nabla w \cdot \nabla w \, dx = \int_D (-\nabla F) \cdot \nabla w \, dx = (-\nabla F, \nabla w) \leq$$

$$||\nabla F||_{L^2(D)} ||\nabla w||_{L^2(D)} \leq ||F||_{H^1(D)} ||w||_{H^1(D)}$$

Since D is bounded we also have due to Poincaré's inequality

$$\int_D |\nabla w|^2 \, dx = \frac{1}{2} \int_D |\nabla w|^2 \, dx + \frac{1}{2} \int_D |\nabla w|^2 \, dx$$

$$\geq \frac{c}{2} \int_D |w|^2 \, dx + \frac{1}{2} \int_D |\nabla w|^2 \, dx \geq c||w||^2_{H^1(D)}$$

Finally we get

$$c||w||^2_{H_1(D)} \leq \int_D |\nabla w|^2 \, dx \leq ||F||_{H^1(D)} ||w||_{H^1(D)} \Rightarrow ||w||_{H^1(D)} \leq c||F||_{H^1(D)}$$

This gives us the bound on u

$$||u||_{H^1(D)} = ||w+F||_{H^1(D)} \leq ||w||_{H^1(D)} + ||F||_{H^1(D)} \leq c||F||_{H^1(\partial D)} \leq c||f||_{H^{1/2}(\partial D)}$$

We see that by differentiating the equations we can get the same bounds on the derivatives of u. In fact it is also a classical result that if ∂D is bounded and in C^k where $s \leq k$

$$||u||_{H^s(D)} \leq c||f||_{H^{s-1/2}(\partial D)} \tag{A.4}$$

see [12].

A.7.2 The Stationary Navier - Stokes Equations

The homogenous case

In this section we will state the weak formulation of the Stokes and Navier-Stokes equations. A big advantage of using the weak formulations is that, as we shall see, the pressure disappears from the equations. Also as in the Laplace case, we only require that the solution is in H_0^1. The homogenous stationary Navier-Stokes equations are stated as follows.

$$\begin{cases} -\nu\Delta u + \sum_{i=1}^{n} u_i \partial_i u + \nabla p = f & in\ D \\ \operatorname{div} u = 0 & in\ D \\ u = 0 & on\ \partial D \end{cases}$$

Note that u is now a vector valued function, which means that the first equation is in fact a system of equations. We define our space of solutions to be

$$V = \{u \in H_0^1(D), \operatorname{div}\ u = 0\}$$

We now multiply the first equation with a function $\phi \in V$ and integrate.

$$\sum_k \left(\int_D (-\nu\Delta u_k \phi_k)\, dx + \int_D \sum_i u_i(\partial_i u_k)\phi_k\, dx + \int_D (\partial_k p)\phi_k\, dx \right) =$$

$$\sum_k \int_D f_k \phi_k\, dx \tag{A.5}$$

If we apply Gauss theorem to the first term in (A.5) we get

$$\int_D (-\nu\Delta u_k \phi_k)\, dx = -\nu \int_D \sum_i (\partial_{ii} u_k)\phi_k\, dx = -\int_D \sum_i \Big(\partial_i((\partial_i u_k)\phi_k) - (\partial_i u_k)(\partial_i \phi_k) \Big)\, dx =$$

$$-\int_{\partial D} \sum_i (\partial_i u_k)\phi_k n_i dS + \int_D (\nabla u_k)\phi_k\, dx$$

For the pressure term in (A.5) we get

$$\sum_k \int_D \partial_k p \phi_k\, dx = \sum_k \left(\int_D \partial_k (p\phi_k)\, dx - \int_D p\partial_k \phi_k\, dx \right) =$$

$$\sum_k \int_{\partial D} p\phi_k n_k dS - \int_D p \operatorname{div} \phi\, dx$$

The second term disappear since $\phi \in V$. For the nonlinear term we get

$$\int_D \sum_i u_i(\partial_i u_k)\phi_k\, dx = \int_D \sum_i (\partial_i(u_i u_k)\phi_k - (\partial_i u_i) u_k \phi_k)\, dx =$$

$$\int_D \left(\sum_i (\partial_i(u_i u_k \phi_k) - u_i u_k \partial_i \phi_k) - u_k \phi_k \operatorname{div} u \right)\, dx =$$

$$\int_{\partial D} \sum_i u_i u_k \phi_k n_i dS - \int_D \sum_i u_i u_k \partial_i \phi_k\, dx$$

We now collect the boundary terms

$$\int_{\partial D} \left(\sum_i (-\nu(\partial_i u_k) n_i + u_i u_k n_i) + p n_k \right) \phi_k dS$$

These terms all disappears since ϕ is zero on ∂D, therefore they will have to be specified as boundary terms. They are called the natural boundary conditions for Navier-Stokes equations, and can be written as

$$-\nu\frac{\partial u_k}{\partial n} + (u \cdot n)u_k + pn_k = g_k \text{ on } \partial D$$

The weak formulation of Navier-Stokes will now be: Find $u \in V$ such that

$$\nu \int_D \nabla u_k \cdot \nabla \phi_k \, dx - \int_D u_k(u \cdot \nabla \phi_k) \, dx = \int_D f_k \phi_k \, dx$$

for all $\phi \in V$. An alternative way to formulate the problem (used in [7]) is to keep the nonlinear term as in (A.5). The problem with this is that the expression $\sum u_i \partial_i u \phi$ does not necessarily make sense for any u, and ϕ in V. To avoid this problem we introduce a new space

$$\tilde{V} = \{u \in H_0^1(D) \cap L^n(D), \text{div } u = 0\}$$

equipped with the norm

$$||u||_{\tilde{V}} = ||u||_{H_0^1(D)} + ||u||_{L^n(D)}$$

Due to [7] we have

$$\left| \int_D \sum_i u_i(\partial_i v_k)\phi_k \, dx \right| \leq c(n)||u||_{H_0^1(D)}||v||_{H_0^1(D)}||u||_{\tilde{V}}$$

which means that the nonlinear term is well defined. In general $\tilde{V}$ is a subspace of V however if n is 2 or 3 then $\tilde{V} = V$. Therefore we get the alternative weak formulation:

Find $u \in V$ such that

$$a(u, \phi) + b(u, u, \phi) = (f, \phi) \tag{A.6}$$

for all $\phi \in V$. Where

$$a(u, \phi) = \nu \int_D \nabla u \cdot \nabla \phi \, dx$$

and

$$b(u, v, w) = \int_D \sum_i u_i \partial_i v w \, dx$$

The weak formulation of the Stokes equations are the same except that there is no nonlinear term. Due to [7](p.35 proposition 2.3) we also have a very useful regularity result. If u and p solves the Stokes equations

$$\begin{cases} -\nu\Delta u + \nabla p = f & \textit{in } D \\ \text{div } u = g & \textit{in } D \\ u = h & \textit{on } \partial D \end{cases}$$

(in $\mathbb{R}^2$ or $\mathbb{R}^3$) and m integer ≥ -1 then

$$||u||_{H^{m+2}(D)} + ||p||_{H^{m+1}(D)/\mathbb{R}} \leq c(||f||_{H^m(D)} + ||g||_{H^{m+1}(D)} + ||h||_{H^{m+3/2}(D)}) \tag{A.7}$$

The non homogenous case

Because of the nonlinear term the non homogenous version of Navier stokes equations can not be transformed directly into the homogenous via trace theorems. However something similar can be done. In this section we will derive a regularity bound for the non homogenous version of Navier-Stokes equations which will be needed to show that if the boundary data f is small then u will also be small. We have

$$\begin{cases} -\nu\Delta u + \sum_{i=1}^{n} u_i\partial_i u + \nabla p &= 0 \ D \\ \text{div } u &= 0 \ \ D \\ u &= f \ \ \partial D \end{cases}$$

To be able to transform this to a problem similar to the homogenous problem we need to find a vector field which takes the same values as u on the boundary and has divergence zero. We can do this by solving the following Stokes problem.

$$\begin{cases} -\Delta\xi + \nabla r &= 0 \quad \textit{in } D \\ \text{div } \xi &= 0 \quad \textit{in } D \\ \xi &= f \quad \textit{on } \partial D \end{cases}$$

(A.7) now gives us that

$$||\xi||_{H^1(\Phi(D))} \leq c_0||f||_{H^{1/2}(D)} \tag{A.8}$$

If we put $\tilde{u} = u - \xi$ we get that $\tilde{u}$ solves

$$\begin{cases} -\nu\Delta\tilde{u} + \sum_{i=1}^{n} \tilde{u}_i\partial_i\tilde{u} + \sum_{i=1}^{n} \tilde{u}_i\partial_i\xi + \sum_{i=1}^{n} \xi_i\partial_i\tilde{u} + \nabla p &= \tilde{f} \ D \\ \text{div } \tilde{u} &= 0 \ \ D \\ \tilde{u} &= 0 \ \ \partial D \end{cases}$$

where $\tilde{f} = \Delta\xi - \sum_i \xi_i\partial_i\xi$. In the weak formulation we get

$$a(\tilde{u},\chi) + b(\tilde{u},\tilde{u},\chi) + b(\tilde{u},\xi,\chi) + b(\xi,\tilde{u},\tilde{u}) =< \tilde{f},\tilde{u} > \tag{A.9}$$

for all χ in V, with a and b defined as in the previous section. One can prove in the almost the same way as for the homogenous Navier-Stokes that this equation has a solution and that it is unique for low Reynolds number (see [7]). We can also prove a regularity result for the solution. Putting $\chi = \tilde{u}$ yields

$$a(\tilde{u},\tilde{u}) + b(\tilde{u},\tilde{u},\tilde{u}) + b(\tilde{u},\xi,\tilde{u}) + b(\xi,\tilde{u},\tilde{u}) =$$

$$\nu||\tilde{u}||^2_{H_0^1(D)} + b(\tilde{u},\xi,\tilde{u}) =< \tilde{f},\chi > \tag{A.10}$$

this gives us the bound

$$\nu||\tilde{u}||^2_{H_0^1(D)} \leq ||\tilde{f}||_{V'(D)}||\tilde{u}||_{H_0^1(D)} + c(n)||\tilde{u}||^2_{H_0^1(D)}||\xi||_{H^1(D)} \Leftrightarrow$$

$$(\nu - c(n)||\xi||_{H^1(D)})||\tilde{u}||_{H_0^1(D)} \leq ||\tilde{f}||_{V'(D)} \tag{A.11}$$

where V' is the dual space of V. Now if we e.g have that

$$c(n)||\xi|| \leq \nu/2 \tag{A.12}$$

then we get our regularity result

$$||\tilde{u}||_{H_0^1(D)} \leq \frac{2}{\nu}||\tilde{f}||_{V'(D)} \tag{A.13}$$

We also have

$$||\tilde{f}||_{V'(D)} = \sup_{||\chi||_{H_0^1(D)}=1} < \tilde{f}, \chi >$$

where χ is in V, and

$$< \tilde{f}, \chi > = < \nu\Delta\xi - \sum_i \xi_i \partial_i \xi, \chi > = -\nu \int_D \nabla\xi \cdot \nabla\chi dx - b(\xi, \xi, \chi) \leq$$

$$\nu||\xi||_{H^1(D)}||\chi||_{H_0^1(D)} + c(n)||\xi||^2_{H^1(D)}||\chi||_{H_0^1(D)} \tag{A.14}$$

(A.14) and (A.13) gives us that

$$||\tilde{u}||_{H_0^1(D)} \leq \frac{2}{\nu}(\nu||\xi||_{H^1(D)} + c(n)||\xi||^2_{H^1(D)}) =$$

$$2||\xi||_{H^1(D)} + \frac{c(n)}{\nu}||\xi||^2_{H^1(D)} \tag{A.15}$$

we now get

$$||u||_{H^1(D)} = ||\tilde{u} + \xi||_{H^1(D)} \leq ||\tilde{u}||_{H^1(D)} + ||\xi||_{H^1(D)} \leq$$

$$k||\tilde{u}||_{H_0^1(D)} + ||\xi||_{H^1(D)} =$$

$$(2k+1)||\xi||_{H^1(D)} + \frac{c(n)2k}{\nu}||\xi||^2_{H^1(D)} \tag{A.16}$$

where $c(n)$ depends on the dimension and k depends on U. (A.8) now gives us

$$||u||_{H^1(D)} \leq (2k+1)c_0||f||_{H^{1/2}(D)} + \frac{c(n)2kc_0^2}{\nu}||f||^2_{H^{1/2}(D)} \tag{A.17}$$

The condition $c(n)||\xi||_{H^1(D)} \leq \nu/2$ will be fulfilled for low Reynolds numbers. By (A.8) we have that (A.12) is fulfilled if

$$c(n)c_0||f||_{H^{1/2}(D)} \leq \frac{\nu}{2} \Rightarrow$$

$$Re := \frac{||f||_{H^{1/2}(\partial D)}}{\nu} \leq \frac{1}{2c(n)c_0} \tag{A.18}$$

Substituting $\frac{||f||_{H^{1/2}(\partial D)}}{\nu}$ for $\frac{1}{2c(n)c_0}$ in (A.17) gives

$$||u||_{H^1(D)} \leq (3k+1)c_0||f||_{H^{1/2}(D)} \tag{A.19}$$

Thus we can conclude that

Lemma A.4 *There exists a constant Re_0 which depends on the dimension and U such that for all*

$$Re := \frac{||f||_{H^{1/2}(\partial D)}}{\nu} \leq Re_0$$

the bound (A.19) *holds.*

A.7.3 Some properties of the term $b(u, v, w)$

From [7] we also have the following very useful properties.

Lemma A.5 *For any open set D in $n = 2$ or 3,*

$$
\begin{aligned}
b(u, v, v) &= 0 && \text{(A.20)}\\
b(u, v, w) &= -b(u, w, v) && \text{(A.21)}
\end{aligned}
$$

$\forall u \in H^1$ *with* $\operatorname{div} u = 0$ *and* $v, w \in H_0^1(D)$.

Proof. We take $u \in V$ and $v \in \mathcal{D}(D)$. For such u and v we have

$$\int_D u_i \partial_i v_j v_j \, dx = \int_D u_i \partial_i \frac{(v_i)^2}{2} \, dx = -\frac{1}{2} \int_D \partial_i u_i (v_j)^2 \, dx$$

$$\Rightarrow b(u, v, v) = -\frac{1}{2} \sum_j \int_D \operatorname{div}\ u(v_j)^2 \, dx = 0$$

By density this is also true for $v \in H_0^1(D)$. For the second property we replace v by $v + w$ and use the multi linear properties of b.

$$0 = b(u, v + w, v + w) = b(u, v, v + w) + b(u, w, v + w) =$$

$$b(u, v, v) + b(u, v, w) + b(u, w, v) + b(u, w, w) = b(u, v, w) + b(u, w, v)$$

A.8 Some properties of $\mathcal{A}$

In this section we establish some simple properties of the mappings in $\mathcal{A}$ which are needed in chapter 3.

A.8.1 Invertibility

Now we will derive some sufficient conditions for invertibility. We need to have this because we need to know that the limit of every sequence in $\mathcal{A}$ is invertible. We will sketch the proof of the following lemma

Lemma A.6 *If $\Phi : U \mapsto U$, Φ is continuous and $\Phi(x) = x$ if $x \in \partial U$ then Φ is invertible if*

$$||\Phi' - I||_2 = \sup_{|v|_2 = 1} |(\Phi' - I)v| \leq k < 1 \quad \forall x \in U \tag{A.22}$$

It is well known that for any Φ which is bounded in $W^{2,\infty}$ norm we can choose a representative which is in $C^{1,1}$, that is we can think of Φ and its derivatives as Lipchitz continuous. We have

$$(\Phi(x) - \Phi(y), x - y) = (\int_0^1 (\Phi'(x - t(y + x)))(x - y) dt, x - y) \tag{A.23}$$

Here $(\cdot,\cdot)$ denotes the standard inner product in $\mathbb{R}^n$ and $|\cdot|_2$ the length of a vector in $\mathbb{R}^n$. We put $E = \Phi - I$. (A.23) then becomes

$$(\Phi(x) - \Phi(y), x - y) = ((x - y) + \int_0^1 (E(x - t(y + x)))(x - y)dt, x - y) =$$

$$|x - y|_2^2 + (\int_0^1 (E(x - t(y + x)))(x - y)dt, x - y) \geq$$

$$|x - y|_2^2 - |(\int_0^1 (E(x - t(y + x)))(x - y)dt, x - y)| \qquad \text{(A.24)}$$

we see that if the matrix norm of E fulfills

$$||E||_2 = \sup_{|v|_2=1} |Ev|_2 \leq k < 1 \quad \forall x \in U \qquad \text{(A.25)}$$

then (A.24) becomes

$$(\Phi(x) - \Phi(y), x - y) \geq (1 - k)|x - y|_2^2 \qquad \text{(A.26)}$$

Applying Cauchy-Schwartz inequality to (A.26) and dividing by $|x - y|_2$ now yields

$$|\Phi(x) - \Phi(y)|_2 \geq (1 - k)|x - y|_2 \qquad \text{(A.27)}$$

(A.27) tells us that Φ will be injective if $||\Phi - Id||_2 \leq k$. For the invertibility we also need to have subjectivity. This can be proven in the following way. It is well known that if Φ is continuous then any simply connected domain U is mapped to a simply connected domain $\Phi(U)$. Simply connected means that if we take any closed curve in the domain it can be continously deformed into a point without passing any points which is not in U. Now assume that $\Phi : U \mapsto U$ is not surjective. If we know that $\Phi(\partial U) = \partial U$ this means that there is a point x_0 inside of U that does not belong to the $\Phi(U)$. However then $\Phi(\partial U)$ can not be deformed to a point without x_0, and therefore $\Phi(U)$ can not be simply connected, which contradicts our assumption.

A.8.2 The pullback operators Φ and Φ^{-1}

The following subsection is due to [1]. Even though [1] define the pullback operators a little bit different than we do, the same proofs can be used.

Definition A.5 *Let D, Ω be open sets in $\mathbb{R}^n$. A bijection $\Phi : D \to \Omega$ is called a k-diffeomorphism if, Φ and Φ^{-1}*

1. *Φ and Φ^{-1} are continuous on $\overline{D}$ and $\overline{\Omega}$, respectively,*
2. *their derivatives of order 1 through k are bounded in $W^{2,\infty}$ on $\overline{D}$ and $\overline{\Omega}$, respectively.*

The pullback operators are defined by $\Phi^* u = u \circ \Phi$ and $(\Phi^{-1})^* u = u \circ \Phi^{-1}$.

We will use the conditions of section A.8.1 for the invertibility.

Lemma A.7 *If Φ and Υ are s-diffeomorphisms then*

$$||Id - \Upsilon \circ \Phi^{-1}||_{W^{s,\infty}(U)} \leq c||\Phi - \Upsilon||_{W^{s,\infty}(U)}$$

where s is a positive integer and c depends only on U and the dimension.

Proof. Since Φ and Υ are in $\mathcal{A}$, we have that $||\Phi||_{W^{2,\infty}(U)}$ and $||\Upsilon||_{W^{2,\infty}(U)}$ are bounded.

$$||Id - \Upsilon \circ \Phi^{-1}||_{L^\infty(U)} = \underset{x \in U}{ess.sup} \; |Id - \Upsilon \circ \Phi^{-1}| =$$

$$\underset{\Phi(\tilde{x}) \in U}{ess.sup} \; |\Phi - \Upsilon \circ \Phi^{-1} \circ \Phi| = ||\Phi - \Upsilon||_{L^\infty(U)}$$

We also have that

$$||x_i - \Upsilon_i \circ \Phi^{-1}||_{L^\infty(U)} = ||\Phi_i \circ \Phi^{-1} - \Upsilon_i \circ \Phi^{-1}||_{L^\infty(U)} =$$

$$||(\Phi_i - \Upsilon_i) \circ (\Phi^{-1})||_{L^\infty(U)} = ||w \circ \Phi^{-1}||_{L^\infty(U)}$$

where $w = \Phi_i - \Upsilon_i$. If we differentiate we get

$$||\partial_i (w \circ \Phi^{-1})||_{L^\infty(U)} = ||\sum_k (\partial_k w) \circ \Phi^{-1} \partial_i \Phi_k^{-1}||_{L^\infty(U)} \leq$$

$$c \sum_k ||\partial_k w||_{L^\infty(U)} \leq c||\Phi - \Upsilon||_{W^{1,\infty}(U)}$$

where c depends on the derivatives of Φ^{-1}. We see that if we differentiate once more we will get

$$||\partial_{ij}(w \circ \Phi^{-1})||_{L^\infty(U)} \leq c||\Phi - \Upsilon||_{W^{2,\infty}(U)}$$

which proves the lemma.

We can of course continue to differentiate and get bounds for higher norms, if Φ and Υ are bounded in those norms. Then the following lemma is true

Lemma A.8 *If Φ and Υ are s-diffeomorphisms then*

$$||Id - \Upsilon \circ \Phi^{-1}||_{W^{s,\infty}(U)} \leq c||\Phi - \Upsilon||_{W^{s,\infty}(U)}$$

where s is a positive integer and c depends only on U and the dimension.

The following Corollary can be proven in exactly the same way as we have done above.

Corollary A.1 *If Φ is an k-diffeomorphism then, the pullback operators Φ^* and $(\Phi^{-1})^*$ are bounded linear mappings from $H^k(\Omega)$ to $H^k(D)$ and from $H^k(D)$ to $H^k(\Omega)$ respectively.*

A.8.3 The term $\det \Psi'$

In order to be able to say something like $u_\Phi \circ \Phi \to u$ when $\Phi \to Id$, we need to have a bound on the term $\det \Psi'$. It seems reasonable to assume that if Φ is close to Id then $\det \Psi'$ is close to 1. We start by looking at $\det \Phi'$.
Due to [2] we have that the determinant is linear in all columns (or rows). Let A and B be two $n \times n$ matrixes with columns $A_1, ..., A_n$ and $B_1, ..., B_n$. We then have

$$\det A - \det B = \det(A_1, A_2, ..., A_n) - \det(B_1, B_2, ..., B_n) =$$

$$\det(A_1 - B_1, A_2, ..., A_n) + \det(B_1, A_2, ..., A_n) - \det(B_1, B_2, ..., B_n) = ... =$$

$$\det(A_1 - B_1, A_2, ..., A_n) + \det(B_1, A_2 - B_2, ..., A_n) + \det(B_1, B_2, ..., A_n - B_n)$$

For a general $n \times n$ matrix we have that $|A_i| \leq ||A||_{L^\infty(D)}$ which gives

$$|| \det A - \det B||_{L^\infty(U)} \leq ||A - B||_{L^\infty(U)} \sum_{k=0}^{n-1} ||A||^{n-1-k}_{L^\infty(U)} ||B||^k_{L^\infty(U)}$$

If we replace A by Φ' and B by I we get

$$|| \det \Phi' - \det I||_{L^\infty(U)} \leq ||\Phi' - I||_{L^\infty(U)} \sum_{k=0}^{n-1} ||\Phi'||^{n-1-k}_{L^\infty(U)}$$

If Φ is in $\mathcal{A}$ we have that the sum is less than a constant which gives us

Lemma A.9 *If Φ is an 1-diffeomorphism then*

$$|| \det \Phi' - \det I||_{L^\infty(U)} \leq c||\Phi - Id||_{W^{1,\infty}(U)}$$

where c depends on $||\Phi||_{W^{1,\infty}(U)}$ and the dimension n

We now look for a bound on $\det \Psi'$. We take $\tilde{x}_j \in D$ and $x \in \Phi(D)$ such that $\tilde{x} = \Psi(x)$ and $x = \Phi(\tilde{x})$. Since $\Phi \circ \Psi = Id$ we have due to the chain rule

$$(\Phi \circ \Psi(x))' = (\Phi' \circ \Psi(x))\Psi(x) = \Phi'(\tilde{x})\Psi'(x) = I \tag{A.28}$$

This gives us that

$$\det \Psi' \det \Phi' = 1 \Rightarrow \det \Psi' = \frac{1}{\det \Phi'}$$

If Φ fulfills

$$\frac{1}{\alpha} \leq |det \Phi'| \leq \alpha$$

(A.28) then we have the following lemma

Lemma A.10 *If $\frac{1}{\alpha} \leq || \det \Phi'||_{L^\infty(U)} \leq \alpha$ then*

$$\frac{1}{\alpha} \leq || \det \Psi'||_{L^\infty(U)} \leq \alpha$$

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